Coordinate Frames for Downward Continuation of Wide Offset Data

Our previous work had not provided a really practical technique for use with wide-offset seismic data. The best handle we had on it was "Slant Frames", SEP March 1974, page 279. The present chapter successfully took us over the hurdle and we now feel comfortable with the widest angles encountered in practical velocity analysis work.

J.F.C. 24 September 1974

Slanted Beam Coupling

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Previously two equations were obtained in order to extrapolate uncoupled downgoing (D) and upcoming (U) waves in the case of slant frames. Now we shall try to couple both equations at the reflector's locations obtaining, therefore, an expression for the reflected waves.

So, let's start from the wave equation:

$$P_{xx} + P_{zz} - \frac{1}{\tilde{v}^2(x,z)} P_{tt} = 0 \quad , \tag{0-1}$$

and the downgoing and upcoming wave transformations:

$$X = x - (\bar{v} \sin\theta) t \tag{2-2a}$$

$$Z = z$$

$$T = -(\sin\theta/\bar{v}) x \pm (\cos\theta/\bar{v}) z + t \quad , \tag{0-2c}$$

where the "+" sign corresponds to the upcoming transformation and the "−" sign to the downgoing transformation. Furthermore we are going to use the reference frames $(X,Z,T) \equiv (x',z',t')$ for the downgoing transformed variables and $(X,Z,T) \equiv (x'',z'',t'')$ for the upcoming ones.

I Uncoupled Equations:

Just for reference, let's compute the uncoupled equations for U and D . If we make

$$P(x,z,t) = D'(x',z',t') \tag{I-1}$$

and transform the wave equation (0-1) according to the downgoing transformation in (0-2), we will have the following relations for the transformed derivatives:

$$P_{xx} = D'_{x'x'} + (\sin^2\theta / \bar{v}^2)D'_{t't'} - (2 \sin\theta / \bar{v})D'_{x't'} \qquad \text{(I-2a)}$$

$$P_{zz} = D'_{z'z'} + (\cos^2\theta / \bar{v}^2)D'_{t't'} - (2 \cos\theta / \bar{v})D'_{z't'} \qquad \text{(I-2b)}$$

$$P_{tt} = (\bar{v}^2 \sin^2\theta)D'_{x'x'} + D'_{t't'} - (2 \bar{v} \sin\theta)D'_{x't'} \quad . \qquad \text{(I-2c)}$$

Replacing them into (0-1) we have:

$$D'_{z'z'} + \left(1 - \frac{\bar{v}^2}{\tilde{v}^2} \sin^2\theta\right)D'_{x'x'} + \frac{1}{\bar{v}^2}\left(1 - \frac{\bar{v}^2}{\tilde{v}^2}\right)D'_{t't'} - \frac{2\sin\theta}{\bar{v}}\left(1 - \frac{v^2}{\tilde{v}^2}\right)D'_{x't'}$$

$$- \frac{2\cos\theta}{\bar{v}} D'_{z't'} = 0 \quad ,$$

which after defining

$$\varepsilon = (\bar{v}^2/\tilde{v}^2) - 1 \quad , \qquad \text{(I-3)}$$

becomes

$$D'_{z't'} = (\bar{v}/2)\cos\theta(1-\varepsilon\tan^2\theta)D'_{x'x'} - (\sec\theta/2\bar{v})\varepsilon D'_{t't'} + \tan\theta\, \varepsilon\, D'_{x't'} + \\ + (\bar{v}\sec\theta/2)D'_{z'z'} \qquad \text{(I-4)}$$

If now we estimate $D'_{z'z'}$ from the previous equation integrating it by t' and differentiating by z' , after assuming $D'_{z'z'z'} \approx 0$, $\varepsilon_{z'} \approx 0$ and $D^{t'}_{x'x'z'} \approx 0$, we get

$$(\bar{v} \sec\theta/2)D'_{z'z'} \cong - (\sec^2\theta/4)\varepsilon\, D'_{z't'} + (\bar{v} \sec\theta\tan\theta/2)\varepsilon\, D'_{x'z'} \; . \qquad \text{(I-5)}$$

Replacing this estimate back into equation (I-4), we finally obtain:

$$\left(1 + \frac{\sec^2\theta}{4}\varepsilon\right)D'_{z't'} = \frac{\bar{v}}{2}\cos\theta(1-\varepsilon\tan^2\theta)D'_{x'x'} - \frac{\sec\theta}{2\bar{v}}\varepsilon D'_{t't'} + \frac{\bar{v}\sec\theta\tan\theta}{2}\varepsilon\, D'_{x'z'}$$

$$+ \tan\theta\, \varepsilon\, D'_{x't'} \qquad \text{(I-6)}$$

For the case $\varepsilon = 0$ $(\bar{v}=\tilde{v})$, equation (I-6) gives the already known result

$$D'_{z't'} = (\bar{v} \cos\theta / 2)D'_{x'x'} \quad , \qquad \text{(I-7)}$$

while for $\theta = 0°$ (normal incidence) it gives:

$$(1 + \varepsilon/4) D'_{z't'} = (\bar{v}/2) \quad D'_{x'x'} - (\varepsilon/2\bar{v}) D'_{t't'} \tag{I-8}$$

Following the same steps but using the upcoming transformations in (0-2), we get for the upcoming transmitted waves:

$$U''_{z''t''} = -\frac{\bar{v}}{2}\cos\theta(1-\varepsilon\tan^2\theta)U''_{x''x''} + \frac{\sec\theta}{2\bar{v}}\varepsilon U''_{t''t''} - \tan\theta\,\varepsilon\,U_{x''t''} - \frac{\bar{v}\sec\theta}{2}U''_{z''z''} \tag{I-9}$$

and

$$(1+\frac{\sec^2\theta}{4}\varepsilon)U''_{z''t''} \cong -\frac{\bar{v}}{2}\cos\theta(1-\varepsilon\tan^2\theta)U''_{x''x''} + \frac{\sec\theta}{2\bar{v}}\varepsilon U''_{t''t''} + \frac{\bar{v}\sec\theta\tan\theta}{2}\varepsilon U''_{x''z''} - \tan\theta\,\varepsilon\,U''_{x''t''} \tag{I-10}$$

II <u>Coupled Equations</u>:

In order to compute reflected waves, we can follow at least two different procedures: either we can compute them from the matrix formulation, describing wave propagation in layered media or we can follow the idea proposed by Claerbout in the "coarse grid calculation" paper. Both procedures give slightly different results in terms of velocity error estimations (for $\varepsilon \neq 0$), that could be evaluated later. We will follow here the second procedure. According to it, making

$$P(x,z,t) = D'(x',z',t') + U''(x'',z'',t'') \tag{II-1}$$

is equivalent to coupling both solutions. Then, in order to get, for example, the equation for the coupled downgoing waves, we transform the wave equation (0-1) with (II-1) according to the downgoing transformation in (0-2) and later subtract off the equation corresponding to the upcoming transmitted waves (I-10).

If we carefully do the above described, after dropping $D'_{z'z'}$, we will end with the following equation:

$$D'_{z't'} = \frac{\bar{v}}{2}\cos\theta(1-\varepsilon\tan^2\theta)D'_{x'x'} - \frac{\sec\theta}{2\bar{v}}\varepsilon D'_{t't'} + \tan\theta\,\varepsilon D'_{x't'} + \frac{\bar{v}\sec\theta}{2}U''_{z''z''}$$

$$- \frac{\sec^2\theta}{4} U_{z''t''} + \frac{\bar{v}\sec\theta\tan\theta}{2}\varepsilon U''_{x''z''} \qquad \text{(II-2)}$$

To obtain a more explicit relation with what could be defined as a reflection coefficient, now we estimate $U''_{z''z''}$ from equation (I-9) in the same way we did for $D'_{z'z'}$ in (I-5). But this time we have to include transmission losses which means that ε_z is no longer $0(\varepsilon_z \neq 0)$. Therefore from (I-9) we will have (assuming again $U''^{t''}_{x''x''z''} = 0$):

$$U''_{z''z''} \cong \frac{\sec\theta}{2}\varepsilon_{z''}\left(\frac{1}{\bar{v}}U''_{t''} - 2\sin\theta\; U''_{x''} - \bar{v}\;\sin^2\theta\; U''^{t''}_{x''x''}\right) +$$

$$+ \frac{\sec\theta}{2}\varepsilon\left(\frac{1}{\bar{v}}U''_{z''t''} - 2\sin\theta\; U''_{x''z''}\right) \qquad \text{(II-3)}$$

Replacing this estimate back into (II-2) the two terms proportional to ε in the coupling part cancel and we are left, as we might have expected already, only with terms proportional to $\varepsilon_{z''}$ in (II-3):

$$D'_{z't'} = \frac{\bar{v}}{2}\cos\theta(1-\tan^2\theta)D'_{x'x'} - \frac{\sec\theta}{2\bar{v}}\varepsilon D'_{t't'} + \tan\theta\,\varepsilon D'_{x't'} +$$

$$+ \frac{\varepsilon_{z''}}{4}\sec^2\theta\left(U''_{t''} - 2\bar{v}\sin\theta\; U''_{x''} - \bar{v}^2\sin^2\theta\, U''^{t''}_{x''x''}\right) \qquad \text{(II-4)}$$

Since we want to express U'' also in the downgoing frame x', z', t' : $U''(x', z', t' + (2\cos\theta/\bar{v})z')$, just notice that:

$$U''_{t''} = U'_{t'} \qquad \text{(II-5a)}$$

$$U''_{x''} = U'_{x'} \qquad \text{(II-5b)}$$

$$U''^{t''}_{x''x''} = U'^{t'}_{x'x'} \qquad \text{(II-5c)}$$

$$\varepsilon_{z''} = \varepsilon_{z'} \qquad \text{(II-5d)}$$

Therefore equation (II-4) finally becomes:

$$D'_{z't'} = \frac{\bar{v}}{2}\cos\theta(1-\varepsilon\tan^2\theta)D'_{x'x'} - \varepsilon\frac{\sec\theta}{2\bar{v}}D'_{t't'} + \varepsilon\tan\theta\, D'_{x't'}$$

$$+ \frac{\varepsilon_{z'}}{4}\sec^2\theta(U''_{t'} - 2\bar{v}\sin\theta\, U''_{x'} - \bar{v}^2\sin^2\theta\, U''^{t'}_{x'x'}) \qquad \text{(II-6)}$$

Following the same steps we can get the corresponding equation for the upcoming reflected waves:

$$U''_{z''t''} = -\frac{\bar{v}}{2}\cos\theta(1- \tan^2\theta)U''_{x''x''} + \varepsilon\frac{\sec\theta}{2\bar{v}}U''_{t''t''} - \varepsilon\tan\theta\, U''_{x''t''} +$$

$$+ \frac{\varepsilon_{z''}}{4}\sec^2\theta(D''_{t''} - 2\bar{v}\sin\theta\, D'_{x''} - \bar{v}2\sin^2\theta\, D'^{t''}_{x''x''}) \qquad \text{(II-7)}$$

where $D' = D'(x'', z'', t'' - (2\cos\theta/\bar{v})z'')$.

It can be shown that once these last equations are written in finite difference notation, the reflection coefficient, otherwise defined through the impedance functions I as $c = (I_{i+1} - I_i) / (I_{i+1} + I_i)$, comes out to be proportional to $-\sec^2\theta\ \varepsilon_z / 4$. So, keeping in mind this proportionality factor, we may as well define:

$$c(x,z,\theta) = -\sec^2\theta\,\frac{\varepsilon_z}{4} . \qquad \text{(II-8)}$$

Replacing back into the equation for the reflected waves (II-7) we then get:

$$U''_{z''t''} = -\frac{\bar{v}}{2}\cos\theta(1 - \varepsilon\tan^2\theta)U''_{x''x''} + \frac{\varepsilon\sec\theta}{2\bar{v}}U''_{t''t''} - \varepsilon\tan\theta U''_{x''t''}$$

$$- c(x,z,\theta)(D''_{t''} - 2\bar{v}\sin\theta\, D''_{x''} - \bar{v}^2\sin^2\theta\, D''^{t''}_{x''x''}) \qquad \text{(II-9)}$$

which for the case $\varepsilon = 0$ $(\bar{v}=v)$ becomes:

$$U''_{z''t''} = -\frac{\bar{v}}{2} \cos\theta \; U''_{x''x''} - c(x,z,\theta)\left(D''_{t''} - 2\bar{v} \sin\theta \; D''_{x''} - \bar{v}^2 \sin^2\theta \; D''^{t''}_{x''x''} \right) \quad \text{(II-10)}$$

while for $\theta = 0°$ (normal incidence) gives the already known result:

$$U''_{z''t''} = -\frac{\bar{v}}{2} U''_{x''x''} + \frac{\varepsilon}{2\bar{v}} U''_{t''t''} - c \; D''_{t''} \quad \text{(II-11)}$$