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Extrapolation of Time-Dependent Waveforms along their Path of Propagation

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Summary

Time-dependent waveforms are commonly extrapolated in space by means of rays and occasionally by means of diffraction integrals. It is possible to extrapolate time-dependent waves in space with a partial differential equation derived from the wave equation. There are stable numerical approximations. An example illustrates a mechanism for 'signal-generated noise' which is consistent with observations.

1. Introduction

When a wave propagates in an inhomogeneous medium the waveform changes. Given that the wave has been observed at a suitable number of points in space we may attempt to solve two types of problems. First we may attempt to ascertain the nature of material inhomogeneity along the wave paths, and second, we may attempt to extrapolate the disturbance back to the source in an attempt to discover the nature of the source. With few exceptions, the methods used during the past decade for doing this kind of geophysical work may be summarized as follows: when wave equations are to be used, separability is achieved by considering cases in which the material inhomogeneity is a function of only one spatial co-ordinate. When higher-dimensional inhomogeneity is so severe that it cannot be ignored, then the wave equations are almost always specialized to ray theory. Ray theory is especially useful when only the travel time is required. Although the amplitude may also be obtained by ray theory it is often of marginal utility because amplitude measurement is made ambiguous by changing waveforms. What we develop in this paper is a finite difference approach to the wave equation which tracks the time dependent waveform of a travelling wave in two-dimensionally inhomogeneous material. This is an extension of earlier work done by one of the authors in the frequency domain. Although the Fourier transform relates time-domain solutions to frequency-domain solutions, there are several compelling practical factors which give impetus to this study. When a waveform is small at certain times of interest and large at times which are not of interest, then a satisfactory approximation to the Fourier integral may be difficult to obtain even if values are obtained with good accuracy at many frequencies: an example is the head wave. Another example occurs in reflection seismology where the most interesting part of the waveform is the late-arriving weak echoes. Another example is when the time function is of long duration but only a small portion of it is of interest; this is usually the case with short-period earthquake seismograms where there is never any hope of interpreting more than the wave packets which come from identifiable phases.

Waves move quickly into a large volume of space. Given that it seems to require about 10 sample points per wavelength to achieve even modest computational accuracy and that a typical computer memory contains 100 000 memory cells, it is evident that some kind of practical limit is attained when a wave emitted by a point source in two dimensions has expanded to a radius of 15 wavelengths. This is grossly inadequate for most geophysical examples of air waves, water waves, and seismic waves. Simplification can often be achieved by not attempting to describe the entire volume V, but only a reduced volume V_1 which surrounds the path from the source to the receiver. If a reduced volume V_1 is to be used, care must be taken to avoid artificial reflections from the sides of the volume V_1. Further economy may be achieved if an even smaller volume, say V_2, moves with a wave packet along a wave path.

2. The differential equation

Let us begin the analytical discussion by transforming the scalar wave equation

$$0 = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} \tag{1}$$

into a co-ordinate frame which translates along the z-axis at the wave speed c. For simplicity in discussion we will make various restricting assumptions. Specialists will recognize that most of these assumptions can be relaxed at the cost of a more complicated development. First, we have obviously chosen the z-axis as being along the path of interest. Second, we will neglect gradients of velocity c while retaining the space variation of c. This is a high-frequency approximation.

Energy which propagates with a component in the positive z-direction in a fixed frame will remain stationary or fall backwards with respect to a frame which translates along z at speed c. Let us choose for the co-ordinate transformation

$$\left.\begin{aligned} x' &= x \\ z' &= ct - z \\ t' &= t \end{aligned}\right\} \tag{2}$$

(see Fig. 1). Since we have chosen z' to be directed opposite to z we will have energy moving with a positive velocity component in either co-ordinate frame. Let P' denote the disturbance in the moving frame. We have

$$P(x,z,t) = P'(x',z',t'). \tag{3}$$

It will be convenient to use subscripts to denote partial derivatives. Obviously $P_x = P'_{x'}$ and

$$P_{xx} = P'_{x'x'}. \tag{4}$$

Also

$$P_z = P'_{x'}x'_z + P'_{z'}z'_z + P'_{t'}t'_z = -P'_{z'}$$

so

$$P_{zz} = P'_{z'z'} \tag{5}$$

and

$$P_t = P'_{x'}X'_t + P'_{z'}z'_t + P'_{t'}t'_t = cP'_{z'} + P'_{t'}$$

so

$$\begin{aligned} P_{tt} &= c(cP'_{z'z'} + P'_{t'z'}) + cP'_{z't'} + P'_{t't'} \\ &= c^2 P'_{z'z'} + 2c\,P'_{z't'} + P'_{t't'}. \end{aligned} \tag{6}$$

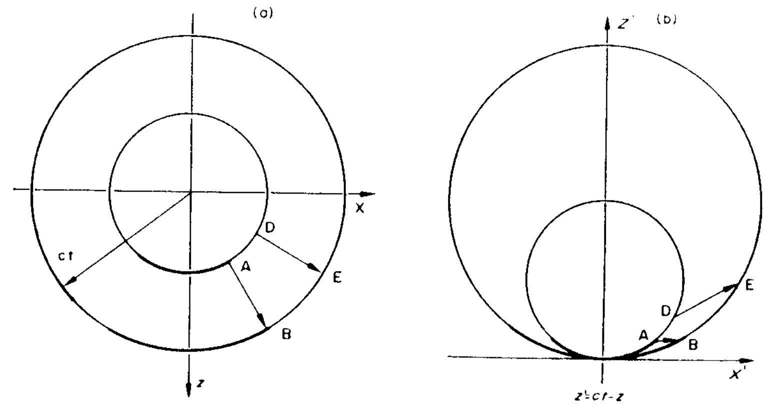

FIG. 1. Expanding spherical wave in fixed co-ordinates (left) and in co-ordinates which translate in the z-direction with the velocity of the wave (right).

Now we may insert (4), (5) and (6) into (1) to obtain

$$0 = P'_{x'x'} - 2c^{-1}P'_{z't'} - c^{-2}P'_{t't'}. \tag{7}$$

Our main interest is with those waves which propagate with approximately the velocity of the new co-ordinate frame. In the moving frame such waves are Doppler-shifted near to zero frequency. This suggests omitting the $P'_{t't'}$ term from (7). Thus (7) becomes

$$P'_{t'z'} = (c/2)P'_{x'x'}. \tag{8}$$

If $\partial/\partial_{t'}$ is replaced by $-i\omega$ then (8) is said to be in the frequency domain. This equation has been studied extensively in the frequency domain by one of the authors (JFC) in a number of earlier papers. Claerbout (1970a) considers the important details relating to computer implementation; Claerbout (1970b) shows the validity of the approximation (8) to be in the range of 20 degrees from the z-axis, and gives a method to extend the range to 90° in concept or 45° at the same computational cost as (8). Claerbout (1971) pays closer attention to space variation of the velocity $c(x,z)$. In this paper we develop a solution directly in the time domain.

To solve equation (8) in a computer the disturbance is defined initially in the $x'-z'$ plane and then time is augmented in steps of $\Delta t'$. A formulation more closely related to observations would be to assume the initial disturbance was measured as a function of x and t and then extrapolate the result along the wave path in the z direction. To achieve this we introduce the change of variables

$$x'' = x \tag{9a}$$

$$z'' = z \tag{9b}$$

$$t'' = t - z/c. \tag{9c}$$

(see Fig. 2).

The new co-ordinate frame stays fixed in space relative to the old one, but time is function of position in the new frame (not unlike the difference between universal time G.M.T. and local solar time). When an observer moves in the $+z = +z''$ direction (west) time will seem to go slower. If he moves at velocity c then time stands

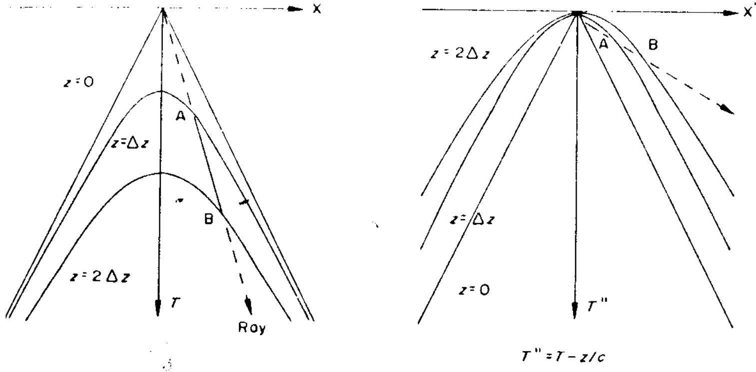

FIG. 2. A point source at $x = 0$, $z = 0$, $t = 0$. Hyperbolas at left indicate arrival times t at $z = 0$, Δz, and $2\Delta z$. When time is a function of position given by $t'' = t - z/c$ the arrival times t'' are as indicated on the right. Energy moves in the direction $+t''$, since on a wave front $z = ct\cos\theta$ and we have

$$t'' = t - z/c = t(1-\cos\theta).$$

still. Referencing time with respect to the time of the earliest possible ray is a great computational convenience because it means that a wave at the source with onset at $t_0 = 0$ will at some distance $z_1 = z_1''$ have its onset at $t_1 = z_1/c$ in the old frame but in the new frame the onset is still at $t_0'' = 0$. This means the wave onset does not move off the finite, perhaps short, computational grid on which the wave packet has been defined. Define the disturbance in the new frame by P'', where

$$P(x,z,t) = P''(x'',z'',t''). \tag{10}$$

Proceeding as before we obtain

$$P_{xx} = P''_{x''x''} \tag{11}$$

$$P_{zz} = P''_{z''z''} - 2c^{-1}P''_{t''z''} + c^{-2}P''_{t''t''} \tag{12}$$

$$P_{tt} = P''_{t''t''}. \tag{13}$$

Inserting these into the wave equation (1) we obtain

$$P''_{t''z''} = (c/2)(P''_{x''x''} + P''_{z''z''}). \tag{14}$$

To see that the last term of (14) is small of higher order for waves travelling at small angles from the z-axis, recall that the solution to the wave equation for waves in the $+z$ direction is an arbitrary function $f(t-z/c) = f''(t'')$. Thus $\partial f''/\partial z''$ vanishes for a wave along the z'' axis. Neglecting $P''_{z''z''}$ we find that (14) reduces to

$$P''_{t''z''} = (c/2)P''_{x''x''} \tag{15}$$

which is the same as equation (8).

3. The difference approximation and a method of solution

Omitting primes from (8) or (15) we have

$$P_{tz} = (c/2)P_{xx}. \tag{16}$$

From a mathematical point of view this equation is completely symmetric with regard to t and z. For the sake of definiteness we will take the point of view of (15), namely, that P was defined at z_0 for all x and t and we are intending to use (16) to extrapolate the waveforms in the z-direction. Obviously we could also take the alternate point of view, that of equation (8). To express (16) in terms of sample time, as we must do for computation, the notion of a Z-transform is essential . A valuable introductory reference to Z-transforms is Treitel & Robinson (1964). First we put (16) into the frequency domain by replacing the time derivative by $-i\omega$.

$$-i\omega P_z = (c/2)P_{xx}. \tag{17}$$

Next we re-express the angular frequency variable ω in terms of the Z-transform variable

$$Z = \exp(i\omega\Delta t).$$

Taking the logarithm we have

$$i\omega\Delta t = \ln Z.$$

Using a well-known expansion for the logarithm we have

$$i\omega\Delta t = 2\left[\frac{Z-1}{Z+1} + \frac{1}{3}\left(\frac{Z-1}{Z+1}\right)^3 + \frac{1}{5}\left(\frac{Z-1}{Z+1}\right)^5 \cdots\right].$$

By retaining only the first term we restrict the validity of our results to waveforms sampled moderately densely (say eight points/wavelength; see Fig. 3). Thus we take

$$-i\omega\Delta t = 2(1-Z)/(1+Z). \tag{18}$$

If P is taken to be a function sampled in the time domain then its Z-transform (which on the unit circle is its Fourier transform) takes the form

$$P = \ldots p_{-1}Z^{-1} + p_0 + p_1 Z + p_2 Z^2 + \ldots. \tag{19}$$

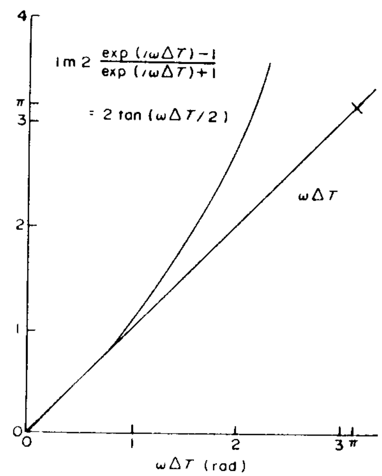

FIG. 3. This shows how the Z-transform approximation forces the data sampling rate to be at least four times the Nyquist sampling rate.

Without loss of generality for the application in mind we may ask P to vanish before $t = 0$, which means that (19) specializes to

$$P = p_0 + p_1 Z + p_2 Z^2 + \ldots = \sum_j p_j Z^j. \tag{20}$$

Putting (18) and (20) into (17) we have

$$\frac{2}{\Delta t} \frac{1-Z}{1+Z} \frac{\partial}{\partial z} P = \frac{c}{2} \frac{\partial^2}{\partial x^2} P. \tag{21}$$

As with all Z-transform equations one has an equation in the frequency domain if one regards Z as taking on all numerical values on the unit circle, and one has an equation at each point in the time domain if one identifies coefficients of various powers of Z. Next let us make the z co-ordinate discrete in equation (21); this follows the approach of earlier papers. Let $P(z)$ be denoted by $P^{n\Delta z}$ or more simply by P^n. Utilizing central differences (21) becomes

$$(1-Z)(P^{n+1} - P^n) = \frac{c\Delta t \Delta z}{4} (1+Z) \frac{\partial^2}{\partial x^2} \frac{P^{n+1} + P^n}{2}. \tag{22}$$

The reader will observe that to avoid defining $P^{n+1/2}$ we have used the average $(P^{n+1} + P^n)/2$. It is shown by Claerbout (1970a) that the more general form $\theta P^{n+1} + (1-\theta) P^n$ (where θ is the implicit/explicit parameter whose value lies between one-half and one) offers some advantage in attenuation of off-axis waves. This is an important parameter is practice but we set $\theta = \frac{1}{2}$ for simplicity since a satisfactory discussion of its use already was given by Claerbout (1970a).

Finally it remains to make (22) discrete with respect to the x-co-ordinate. Let p_j^n for fixed n (depth) and j (time) be called a vector because it contains an unwritten subscript which refers to variation in the x-direction. Then $\partial^2/\partial x^2$ is like a tri-diagonal matrix with the second-difference operator $(1, -2, 1)/\Delta x^2$ on the main diagonal. Denoting by T a tri-diagonal matrix with $(-1, 2, -1)$ on the main diagonal, (22) may be written

$$(1-Z)(P^{n+1} - P^n) = -\frac{c\Delta t \Delta z}{8\Delta x^2} (1+Z)\, T(P^{n+1} + P^n) \tag{23}$$

which we abbreviate by

$$(1-Z)(P^{n+1} - P^n) = -a(1+Z)\, T(P^{n+1} + P^n). \tag{24}$$

We bring terms depending on the disturbance at $(n+1)\,\Delta z$ to the left and the others to the right to get

$$[(1+aT) - Z(1-aT)]\, P^{n+1} = [(1-aT) - Z(1+aT)]\, P^n. \tag{25}$$

Recognizing that P^n and P^{n+1} represent Z-transform polynomials (20) of the disturbance at z and $z + \Delta z$, we may identify in (25) the coefficient of Z^{j+1}. For any $j > 0$ the coefficient is

$$(1+aT)\, p_{j+1}^{n+1} - (1-aT)\, p_j^{n+1} = (1-aT)\, p_{j+1}^n - (1+aT)\, p_j^n \tag{26}$$

which we may rearrange to

$$(1+aT)\, p_{j+1}^{n+1} = (1-aT)\, p_j^{n+1} + (1-aT)\, p_{j+1}^n - (1+aT)\, p_j^n. \tag{27}$$

Suppose for some particular n and j that everything on the right-hand side is known. Computationally the right side then represents a known vector of NX components where NX refers to the number of points which have been sampled on the x-axis. The factor $(1+aT)$ represents a tri-diagonal matrix of size NX. Thus we have a very sparse set of simultaneous equations which may be solved (by the method of Richtmyer & Morton, 1967, 198–201, for example) for the vector p_{j+1}^{n+1}. Of course boundary conditions are required at the ends of the vector. The authors have thus far used zero-slope conditions at the extremes on the x-axis. This is satisfactory when either the function or its x-derivative is sufficiently small at the boundaries.

It is clear that a satisfactory initial condition for the recursion (27) is knowledge of both p_j^0 for all j and p_0^n for all n, both of course for all values of the x-co-ordinate $(k\Delta x)$. Let us consider an example of a point source at $t = 0$ and $z = -10\Delta z$. Clearly $p_0^n = 0$ for all $n \geqslant 0$. Also p_j^0 vanishes for enough values of j for the wave to have time to get to $z = 0$. After that p_j^0 is an arbitrary source waveform. By elementary geometry the x-dependence of p_j^0 must be worked out to conform to the well-known hyperbola in the $x-t$ plane.

4. Stability

In this section we will show that stability is assured for all values of Δx, Δz, and Δt. Stability is lost if the calculation is set up in an unnatural direction. For example, we know that waveforms move in the $+z'$ direction in the x'-z' plane. In other words information at $z' - \Delta z'$ will later be at z'. Thus it is reasonable and stable to calculate $P'(z')$ from present and past values of $P'(z' - \Delta z')$ but it is unreasonable and unstable to try to calculate $P'(z')$ from the present and past values of $P'(z' + \Delta z')$. To get $P'(z')$ from $P'(z' + \Delta z')$ it would be necessary to use present and future values.

First of all the stability of the recursion on x is assured because a is positive and the matrix $(1+aT)$ has $1+2a$ on the main diagonal and $-a$ off the main diagonal, so the matrix is diagonally dominant.

Next let us consider the recursion on time. Eigenvalues for the T matrix may be shown to lie between zero and $+4$. Thus we may consider T in (25) to be replaced by an arbitrary number between zero and $+4$. To determine P^{n+1} from P^n it is necessary to divide (25) by the left-side-polynomial $[(1+aT) - Z(1-aT)]$. This polynomial will be minimum-phase (Treitel & Robinson 1964) because a is positive, T is positive, and the coefficient of Z^0 always dominates the coefficient of Z^1. Notice that the polynomial on the right-hand side of (25) will be definitely not be minimum phase, so that one cannot determine P^n from P^{n+1}.

Finally we show stability with regard to stepping in the z-direction. For this it is satisfactory to show that the transfer function of (25), namely

$$\frac{(1-aT) - Z(1+aT)}{(1+aT) - Z(1-aT)}$$

has unit magnitude for all frequencies (Z on the unit circle) and all horizontal wavelengths $(0 \leqslant T \leqslant 4)$. The transfer function is of the form $(b - cZ)/(c - bZ)$. Since $Z = \exp(i\omega\Delta t)$ we have $Z^{-1} = \exp(-i\omega\Delta t)$. Therefore multiplying the transfer function by its conjugate we have

$$\frac{b-cZ}{c-bZ} \cdot \frac{b-c/Z}{c-b/Z} = \frac{b^2+c^2-bc(Z+1/Z)}{b^2+c^2-bc(Z+1/Z)} = 1.$$

From the point of view of stability with respect to extrapolation in the z direction it is irrelevant whether we go from P^n to P^{n+1} or the reverse. It is the recurrence on time which must go in one particular direction. Finally we remind the reader that z and t, n and j, and ω and k_z (vertical wave number) may be interchanged if they are interchanged *everywhere* throughout the discussion subsequent to equation (16).

5. Example—disturbed plane wave

While most theoretical solutions in wave propagation deal with highly symmetric waves (plane, cylindrical, etc.), nature is seldom so regular. The example shown in Fig. 4 represents a disturbed plane wave which might have been produced as shown in Fig. 5. At the bottom of Fig. 5 a plane atmospheric pressure wave is incident upon a series of circulating cells which tend to advance the wave in some regions while retarding it in other regions, thus producing the waveform shown at the top of Fig. 5. A somewhat similar situation is the phase grating in optics (Goodman 1968, p. 69), where the monochromatic solution is usually obtained at infinity.

The seven frames in Fig. 4 illustrate the subsequent development of the disturbed wave shown in frame No. 1. Note the frames may be thought of interchangeably in either steps of $\Delta t'$ or $\Delta z''$, as has been discussed previously. The most obvious development is that the energy spreads out as one moves down the figure. The single pulse of the top frame has become an extended oscillatory arrival by the last frame. As time goes on, less and less energy is in the first pulse and more and more is in the oscillatory tail. Note: in the figure the gain is adjusted with each frame to give better contrast. Also, as might be expected, the wave onset time, which is a dramatic function of x in the first frame, is nearly independent of x by the last frame. A surprising feature is that although energy moves back from the first arrival (see Figs 1 and 2) a point of constant phase in the wave tail moves forward toward the first arrival (a point of constant phase is marked by an X on the right side of each frame). Another clear feature of the wave tails is that dip (arrival time dependence on x) increases going down a single frame. The phase shift of the two-dimensional focus, which causes doublets to form, is clearly visible at the point in frame No. 2 marked with an A.

In order to represent a disturbance of infinite extent in x on a finite computer grid, we initialized the problem with a periodic disturbance having zero slope at the side boundaries. Zero-slope boundary conditions are then equivalent to infinite periodic extension in x. A value $a = \frac{1}{4}$ was chosen to given an appropriate variation in progressive frames, with each frame in Fig. 4 representing 5 computational iterations. The solution may be rescaled in several ways due to the interdependence in the constant a (see equation (24)) of $c\Delta t$, Δx and Δz. The calculation in Fig. 4 required about 0·5 min on the IBM 360/67.

It might be valuable to consider various data enhancement processes in the light of Fig. 4. In the process called 'beam-steering', observations as in Fig. 4 would be

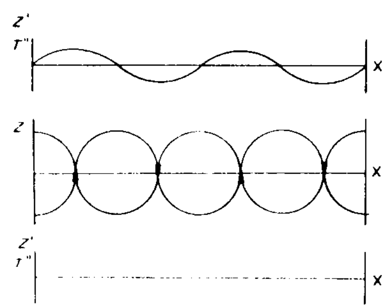

Fig. 5. One means of producing a disturbed plane wave. Incident plane wave at bottom is altered by a material inhomogeneity (centre), resulting in the disturbed wave front at top.

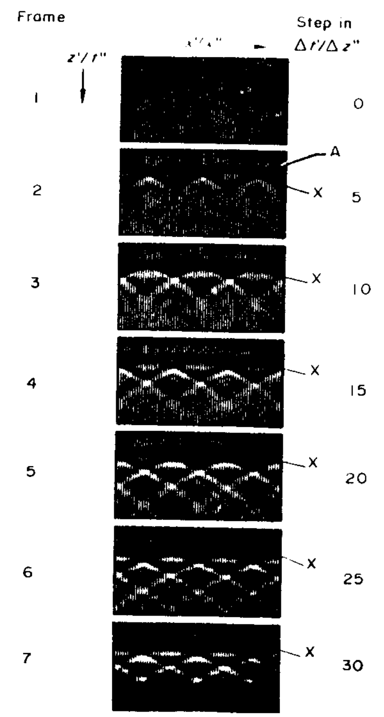

Fig. 4. Disturbed plane wave propagating through a homogeneous space in a moving co-ordinate system. Features of note are: (1) energy moves backwards and toward the sides; (2) the wave onset is a dramatic function of x near the beginning, but by the last frame is nearly independent of x; (3) phase of the wave tail (marked by x in each frame) moves forward; (4) the dip of the wave tail increases in a given frame as one scans down the frame; (5) in frame No. 2, the letter A indicates a two-dimensional focus.

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summed over the x-co-ordinate in an effort to enhance signal and reject noise. Clearly beam-steering will enhance the first arrival while rejecting random noise. What it will also do is to tend to cancel the signal energy which resides in the oscillatory wave tails. If one is really interested in enhancing signal-to-noise ratio it would hardly seem desirable to use a processing scheme which cancels signal energy. As z' or t' is increased the situation becomes increasingly severe since signal energy moves from the initial pulse toward the oscillatory wave tails. The central practical conclusion of this section is that what has often been regarded as 'signal-generated-noise' may turn out to be signal in a potentially valuable form. We can, indeed, expect dramatic results if we are able to learn how to design data enhancement techniques on entire waveforms rather than on the initial pulse alone.

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References

Claerbout, J. F., 1970a. Coarse grid calculations of waves in inhomogeneous media with application to delineation of complicated seismic structure, *Geophysics*, **35**, 407–418.

Claerbout, J. F., 1970b. Numerical holography in *Acoustical Holography*, Vol 3, ed. A. F. Metherell, Plenum Publishing Corp., New York.

Claerbout, J. F., 1971. Toward a unified theory of reflector mapping, *Geophysics*, **36**. 467–481.

Goodman, J. W., 1968. *Introduction to Fourier Optics*, McGraw-Hill Book Co., Inc., New York.

Richtmyer, R. D. & Morton, K. W., 1967. *Difference Methods for Initial Value Problems*, Interscience, New York.

Treitel, S. & Robinson, E. A., 1964. The stability of digital filters, *IEEE Trans. Geoscience Electronics*, **2**, 6–18.