Appendix E

Derivation of the back-projection operator

In section 3.2.2 I formally derived the equation for evaluating the back-projection operator that relates perturbation in the velocity model to perturbations in the beam stacks' offsets. In this appendix I derive the same equation as in the main text [equation (3.11)], but the derivation presented here explicitly shows the relation between the perturbation in beam stacks offset and the movement of reflectors.

For each offset ray parameter p_h and each midpoint ray parameter p_y, the back-projection operator is given by the partial derivatives of the offset h with respect to the velocity model $\mathbf{m}$, evaluated at constant transformed traveltime τ and constant midpoint y. The reflector's position (x, z) must move, so that the transformed traveltime and midpoint remain constant when the velocity is perturbed. These reflector's movements must be taken into account for the offset perturbations to be correctly computed. The desired derivative is evaluated by use of the following formula [equation (3.11) in section 3.2.2],

$$\left.\frac{\partial h}{\partial \mathbf{m}}\right|_{(y,\tau)} = \left.\frac{\delta h}{\delta \mathbf{m}}\right|_{(x,z)} - \left.\frac{\partial h}{\partial \tau}\right|_{(y,\mathbf{m})} \left.\frac{\delta \tau}{\delta \mathbf{m}}\right|_{(x,z)} - \left.\frac{\partial h}{\partial y}\right|_{(\tau,\mathbf{m})} \left.\frac{\delta y}{\delta \mathbf{m}}\right|_{(x,z)}, \tag{E.1}$$

which combines the partial derivatives $\partial h/\partial \tau$ and $\partial h/\partial y$ computed at constant velocity model on the manifold $h(\tau, y)$ defined by ray tracing, with the partial derivatives $\delta h/\delta \mathbf{m}$, $\delta \tau/\delta \mathbf{m}$, and $\delta y/\delta \mathbf{m}$ computed at constant reflector position (x, z). The raypaths derivatives are computed by use of the gradient computations presented in Appendix C.

The modeling by ray tracing establishes a map between the reflector space (x, z) and the data space (τ, y, h). This map is function of the velocity model $\mathbf{m}$ and can be represented by the triplet of functions $[\tau(\mathbf{m}, x, z), y(\mathbf{m}, x, z), h(\mathbf{m}, x, z)]$. Differentiating these functions with respect to the model perturbations $d\mathbf{m}$ and with respect to the reflector movements

(dx, dz), we get the following relations:

$$d\tau = \left.\frac{\delta\tau}{\delta\mathbf{m}}\right|_{(x,z)} d\mathbf{m} + \left.\frac{\partial\tau}{\partial x}\right|_{(z,\mathbf{m})} dx + \left.\frac{\partial\tau}{\partial z}\right|_{(x,\mathbf{m})} dz; \tag{E.2}$$

$$dy = \left.\frac{\delta y}{\delta\mathbf{m}}\right|_{(x,z)} d\mathbf{m} + \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})} dx + \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} dz; \tag{E.3}$$

$$dh = \left.\frac{\delta h}{\delta\mathbf{m}}\right|_{(x,z)} d\mathbf{m} + \left.\frac{\partial h}{\partial x}\right|_{(z,\mathbf{m})} dx + \left.\frac{\partial h}{\partial z}\right|_{(x,\mathbf{m})} dz. \tag{E.4}$$

Given a model perturbation $d\mathbf{m}$, the reflector's movement (dx, dz) is determined by the conditions that the transformed travel time τ and the midpoint y be constant. Therefore, imposing $d\tau = 0$ and $dy = 0$ in equation (E.2) and (E.3), and solving for (dx, dz) gives

$$dx|_{(\tau,y)} = \left(\frac{\left.\frac{\partial\tau}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\delta y}{\delta\mathbf{m}}\right|_{(x,z)} - \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\delta\tau}{\delta\mathbf{m}}\right|_{(x,z)}}{\left.\frac{\partial\tau}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial\tau}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})}} \right) d\mathbf{m}, \tag{E.5}$$

and

$$dz|_{(\tau,y)} = \left(\frac{\left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\delta\tau}{\delta\mathbf{m}}\right|_{(x,z)} - \left.\frac{\partial\tau}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\delta y}{\delta\mathbf{m}}\right|_{(x,z)}}{\left.\frac{\partial\tau}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial\tau}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})}} \right) d\mathbf{m}. \tag{E.6}$$

Substituting these values of (dx, dz) into equation (E.4), and regrouping the terms, we find

$$\begin{aligned} dh|_{(\tau,y)} \quad = \quad & \left.\frac{\delta h}{\delta\mathbf{m}}\right|_{(x,z)} d\mathbf{m} + \\ & \left(\frac{\left.\frac{\partial h}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})} - \left.\frac{\partial h}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})}}{\left.\frac{\partial\tau}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial\tau}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})}} \right) \left.\frac{\delta\tau}{\delta\mathbf{m}}\right|_{(x,z)} d\mathbf{m} + \\ & \left(\frac{\left.\frac{\partial h}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial\tau}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial h}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial\tau}{\partial x}\right|_{(z,\mathbf{m})}}{\left.\frac{\partial\tau}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial\tau}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})}} \right) \left.\frac{\delta y}{\delta\mathbf{m}}\right|_{(x,z)} d\mathbf{m}. \end{aligned} \tag{E.7}$$

This equation express the perturbations in offset caused by perturbations in the velocity model, at constant transformed traveltime and midpoint. The partial derivatives computed at fixed reflector position are multiplied by terms that are evaluated at constant velocity model, and therefore can be computed from the results of ray tracing. These terms can be actually related to the partial derivatives computed on the manifold $h(\tau, y)$. From equation (E.2) and (E.4), assuming $d\mathbf{m} = 0$, we derive,

$$\left.\frac{\partial h}{\partial \tau}\right|_{(y,\mathbf{m})} = \frac{\left.\frac{\partial h}{\partial x}\right|_{(z,\mathbf{m})} dx|_{(y,\mathbf{m})} + \left.\frac{\partial h}{\partial z}\right|_{(x,\mathbf{m})} dz|_{(y,\mathbf{m})}}{\left.\frac{\partial \tau}{\partial x}\right|_{(z,\mathbf{m})} dx|_{(y,\mathbf{m})} + \left.\frac{\partial \tau}{\partial z}\right|_{(x,\mathbf{m})} dz|_{(y,\mathbf{m})}}, \tag{E.8}$$

and from equation (E.3), assuming $dy = 0$,

$$\left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})} dx|_{(y,\mathbf{m})} + \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} dz|_{(y,\mathbf{m})} = 0. \tag{E.9}$$

Substituting equation (E.9) into equation (E.8) we obtain,

$$\left.\frac{\partial h}{\partial \tau}\right|_{(y,\mathbf{m})} = -\left(\frac{\left.\frac{\partial h}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})} - \left.\frac{\partial h}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})}}{\left.\frac{\partial \tau}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial \tau}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})}}\right). \tag{E.10}$$

Therefore the partial derivative $\partial h/\partial \tau$ is equal to the term that multiplies $\delta\tau/\delta\mathbf{m}$ in equation (E.7).

Similarly, from equations (E.3) and (E.4), assuming $d\mathbf{m} = 0$, we derive,

$$\left.\frac{\partial h}{\partial y}\right|_{(\tau,\mathbf{m})} = \frac{\left.\frac{\partial h}{\partial x}\right|_{(z,\mathbf{m})} dx|_{(y,\mathbf{m})} + \left.\frac{\partial h}{\partial z}\right|_{(x,\mathbf{m})} dz|_{(y,\mathbf{m})}}{\left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})} dx|_{(y,\mathbf{m})} + \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} dz|_{(y,\mathbf{m})}}, \tag{E.11}$$

and from equation (E.2), assuming $d\tau = 0$,

$$\left.\frac{\partial \tau}{\partial x}\right|_{(z,\mathbf{m})} dx|_{(\tau,\mathbf{m})} + \left.\frac{\partial \tau}{\partial z}\right|_{(x,\mathbf{m})} dz|_{(\tau,\mathbf{m})} = 0. \tag{E.12}$$

Substituting equation (E.12) into equation (E.11) we get,

$$\left.\frac{\partial h}{\partial y}\right|_{(\tau,\mathbf{m})} = -\left(\frac{\left.\frac{\partial h}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial \tau}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial h}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial \tau}{\partial x}\right|_{(z,\mathbf{m})}}{\left.\frac{\partial \tau}{\partial x}\right|_{(z,\mathbf{m})} \left.\frac{\partial y}{\partial z}\right|_{(x,\mathbf{m})} - \left.\frac{\partial \tau}{\partial z}\right|_{(x,\mathbf{m})} \left.\frac{\partial y}{\partial x}\right|_{(z,\mathbf{m})}}\right). \tag{E.13}$$

Finally substituting equations (E.10) and (E.13) into equation (E.7) we get the desired result,

$$dh\big|_{(\tau,y)} = \frac{\delta h}{\delta \mathbf{m}}\bigg|_{(x,z)} d\mathbf{m} - \frac{\partial h}{\partial \tau}\bigg|_{(y,\mathbf{m})} \frac{\delta \tau}{\delta \mathbf{m}}\bigg|_{(x,z)} d\mathbf{m} - \frac{\partial h}{\partial y}\bigg|_{(\tau,\mathbf{m})} \frac{\delta y}{\delta \mathbf{m}}\bigg|_{(x,z)} d\mathbf{m}. \qquad \text{(E.14)}$$