Appendix C

Ray tracing in a 2-D velocity model

Ray tracing is a basic element in the implementation of the velocity estimation presented in this thesis. It is employed for modeling the beam-stacked data and for computing the back-projection operator. The first part of this appendix presents a method for computing the raypath and traveltime of a fan of rays: it solves a ray-tracing system of first-order ordinary differential equations, derived from the Eikonal equation (Cerveny, 1987). The second part of the appendix solves the more difficult problem of evaluating the derivatives of the ray-path and traveltime with respect to the slowness model. These derivatives are computed by use of a continuation method (Fawcett, 1983).

C.1 COMPUTING THE RAYPATH AND TRAVELTIME

Rays are traced in a general 2-D medium with slowness function $S(z, x; \mathbf{m})$, where z is depth, x the horizontal coordinate, and $\mathbf{m}$ is the vector of slowness parameter. The slowness function is parametrized by use of B-spline functions (Appendix D).

The rays are traced in depth by the solution of the ray-tracing system

$$dx = \frac{p_x dz}{p_z} \tag{C.1a}$$

$$dp_z = \frac{S(z, x, \mathbf{m}) S_z(z, x, \mathbf{m}) dz}{p_z} \tag{C.1b}$$

$$dp_x = \frac{S(z, x, \mathbf{m}) S_x(z, x, \mathbf{m}) dz}{p_z}; \tag{C.1c}$$

and the traveltime is computed with the equation

$$dt = \frac{S^2(z, x, \mathbf{m})dz}{p_z}. \tag{C.1d}$$

Here $S_z(z, x, \mathbf{m})$ and $S_x(z, x, \mathbf{m})$ are the partial derivatives of slowness with respect to the spatial coordinates, p_x is the horizontal ray parameter, and p_z is the vertical ray parameter.

The rays are traced down from the surface, thus the initial conditions are

$$z = 0$$

$$x = x_0$$

$$p_z = \sqrt{S^2(0, x_0, \mathbf{m}) - p_{obs}^2}$$

$$p_x = p_{obs}$$

and

$$t = 0,$$

where p_{obs} is the horizontal ray parameter measured at the surface by beam stacks, that is, the shot ray parameter p_s or the receiver ray parameter p_r.

The ray-tracing system can be numerically solved by use of a standard Runge-Kutta algorithm. The computation can be efficiently vectorized by the simultaneous tracing of a fan of rays (Van Trier, 1988).

C.2 COMPUTING THE DERIVATIVES WITH RESPECT TO THE SLOWNESS MODEL

The velocity estimation needs the derivatives of the raypath with respect to the slowness model, for the evaluation of the linear operator $\mathbf{G}_h$, that relates velocity perturbations to beam stacks' offset variations. These derivatives are computed with a continuation method.

The differential equations in the ray-tracing system can be approximated by the following system of difference equations:

$$X^k(\mathbf{m}) = x^k - x^{k-1} - \frac{\bar{p}_x^k \Delta z}{\bar{p}_z^k} \quad \text{(C.2a)}$$

$$P_z^k(\mathbf{m}) = p_z^k - p_z^{k-1} - \frac{\bar{S}^k \bar{S}_z^k \Delta z}{\bar{p}_z^k} \quad \text{(C.2b)}$$

$$P_x^k(\mathbf{m}) = p_x^k - p_x^{k-1} - \frac{\bar{S}^k \bar{S}_x^k \Delta z}{\bar{p}_z^k} \quad \text{(C.2c)}$$

and

$$T^k(\mathbf{m}) = t^k - t^{k-1} - \frac{(\bar{S}^k)^2 \Delta z}{\bar{p}_z^k}, \quad \text{(C.2d)}$$

where $\Delta z = z^k - z^{k-1}$ is constant; $\bar{S}^k$, $\bar{S}_z^k$, and $\bar{S}_x^k$ are the slowness and its derivatives evaluated at the intermediate point $\left[\bar{z}^k = (z^k + z^{k-1})/2,\ \bar{x}^k = (x^k + x^{k-1})/2\right]$; and $\bar{p}_z^k$ and $\bar{p}_x^k$ are the averages $\bar{p}_z^k = (p_z^k + p_z^{k-1})/2$ and $\bar{p}_x^k = (p_x^k + p_x^{k-1})/2$. The results of ray tracing can be used for precisely evaluating these quantities along the ray.

Equations (C.2) must be satisfied for all values of the slowness model $\mathbf{m}$, therefore the derivatives of $X^k(\mathbf{m})$, $P_z^k(\mathbf{m})$, and $P_x^k(\mathbf{m})$ with respect to the slowness model are zero. These derivatives with respect to the model parameter m_i can be expanded with the chain rule as

$$\frac{\partial X^k}{\partial m_i} = \frac{\partial X^k}{\partial x^k}\frac{\partial x^k}{\partial m_i} + \frac{\partial X^k}{\partial x^{k-1}}\frac{\partial x^{k-1}}{\partial m_i} + \frac{\partial X^k}{\partial \bar{p}_z^k}\frac{\partial \bar{p}_z^k}{\partial m_i} + \frac{\partial X^k}{\partial \bar{p}_x^k}\frac{\partial \bar{p}_x^k}{\partial m_i} = 0 \quad \text{(C.3a)}$$

$$\frac{\partial {P_z}^k}{\partial m_i} = \frac{\partial {P_z}^k}{\partial p_z^k}\frac{\partial {p_z}^k}{\partial m_i} + \frac{\partial {P_z}^k}{\partial p_z^{k-1}}\frac{\partial {p_z}^{k-1}}{\partial m_i} + \frac{\partial {P_z}^k}{\partial \bar{p}_z^k}\frac{\partial \bar{p}_z^k}{\partial m_i} +$$

$$\frac{\partial {P_z}^k}{\partial \bar{S}^k}\frac{\partial \bar{S}^k}{\partial m_i} + \frac{\partial {P_z}^k}{\partial \bar{S}_z^k}\frac{\partial \bar{S}_z^k}{\partial m_i} = 0 \quad \text{(C.3b)}$$

$$\frac{\partial P_x{}^k}{\partial m_i} = \frac{\partial P_x{}^k}{\partial p_x^k}\frac{\partial p_x{}^k}{\partial m_i} + \frac{\partial P_x{}^k}{\partial p_x^{k-1}}\frac{\partial p_x{}^{k-1}}{\partial m_i} + \frac{\partial P_x{}^k}{\partial \bar{p}_z^k}\frac{\partial \bar{p}_z^k}{\partial m_i} +$$

$$\frac{\partial P_x{}^k}{\partial \bar{S}^k}\frac{\partial \bar{S}^k}{\partial m_i} + \frac{\partial P_x{}^k}{\partial \bar{S}_x^k}\frac{\partial \bar{S}_x^k}{\partial m_i} = 0. \tag{C.3c}$$

When all the partial derivatives in equations (C.3) are evaluated it is possible to obtain a system that can be sequentially solved at each step k for the derivatives $\partial x^k/\partial m_i$, $\partial p_z{}^k/\partial m_i$ and $\partial p_x{}^k/\partial m_i$, when the derivatives at the previous step $k-1$ are known. Equations (C.3) are not independent, and therefore a system of three equations in three unknowns must be solved at each step. The three equations can be solved sequentially after these terms are dropped: those that depend on the second derivatives of slowness with respect to the spatial coordinates, and those terms that depend on the product of two first derivatives. It is possible to drop these terms because the slowness function is assumed to be smooth. After these approximations and some simplifications, equations (C.3) become

$$\frac{\partial x^k}{\partial m_i} = \frac{\partial x^{k-1}}{\partial m_i} + \frac{\Delta z}{2\bar{p}_z^k}\left(\frac{\partial p_x{}^k}{\partial m_i} + \frac{\partial p_x{}^{k-1}}{\partial m_i}\right) - \frac{\bar{p}_x^k \Delta z}{2(\bar{p}_z^k)^2}\left(\frac{\partial p_z{}^k}{\partial m_i} + \frac{\partial p_z{}^{k-1}}{\partial m_i}\right) \tag{C.4a}$$

$$\frac{\partial p_z{}^k}{\partial m_i}\left(1 + \frac{\bar{S}^k \bar{S}_z^k \Delta z}{2(\bar{p}_z^k)^2}\right) = \frac{\partial p_z{}^{k-1}}{\partial m_i}\left(1 - \frac{\bar{S}^k \bar{S}_z^k \Delta z}{2(\bar{p}_z^k)^2}\right) +$$

$$\frac{\bar{S}^k \Delta z}{\bar{p}_z^k}\left.\frac{\partial S_z(z,x,\mathbf{m})}{\partial m_i}\right|_{(\bar{z},\bar{x})} + \frac{\bar{S}_z^k \Delta z}{\bar{p}_z^k}\left.\frac{\partial S(z,x,\mathbf{m})}{\partial m_i}\right|_{(\bar{z},\bar{x})} \tag{C.4b}$$

$$\frac{\partial p_x{}^k}{\partial m_i} = \frac{\partial p_x{}^{k-1}}{\partial m_i} + \frac{\bar{S}^k \Delta z}{\bar{p}_z^k}\left.\frac{\partial S_x(z,x,\mathbf{m})}{\partial m_i}\right|_{(\bar{z},\bar{x})} + \frac{\bar{S}_x^k \Delta z}{\bar{p}_z^k}\left.\frac{\partial S(z,x,\mathbf{m})}{\partial m_i}\right|_{(\bar{z},\bar{x})} -$$

$$\frac{\bar{S}^k \bar{S}_x^k \Delta z}{2(\bar{p}_z^k)^2} \left(\frac{\partial {p_z}^k}{\partial m_i} + \frac{\partial {p_z}^{k-1}}{\partial m_i} \right) . \tag{C.4c}$$

Equations (C.4) can be solved sequentially: start from equation (C.4b), follow with equation (C.4c), and then equation (C.4a). The initial conditions at the surface are:

$$\frac{\partial x^0}{\partial m_i} = 0, \tag{C.5a}$$

$$\frac{\partial {p_z}^0}{\partial m_i} = \frac{S(0, x_0, \mathbf{m})}{\sqrt{S^2(0, x_0, \mathbf{m}) - p_{obs}^2}} \left. \frac{\partial S(0, x_0, \mathbf{m})}{\partial m_i} \right|_{(0, x_0)} , \tag{C.5b}$$

$$\frac{\partial {p_x}^0}{\partial m_i} = 0. \tag{C.5c}$$

Once the gradient of the raypath with respect to the slowness model is computed, it is easy to evaluate the gradient of traveltime. Differentiating equation (C.2d), I obtain the equation

$$\frac{\partial t^k}{\partial m_i} = \frac{\partial t^{k-1}}{\partial m_i} + \frac{2\bar{S}^k \Delta z}{\bar{p}_z^k} \left[\left. \frac{\partial S(z, x, \mathbf{m})}{\partial m_i} \right|_{(\bar{z}, \bar{x})} + \frac{\bar{S}_x^k}{2} \left(\frac{\partial x^k}{\partial m_i} + \frac{\partial x^{k-1}}{\partial m_i} \right) \right] -$$

$$\frac{(\bar{S}^k)^2 \Delta z}{2(\bar{p}_z^k)^2} \left(\frac{\partial {p_z}^k}{\partial m_i} + \frac{\partial {p_z}^{k-1}}{\partial m_i} \right) , \tag{C.6}$$

which can be used to evaluate sequentially the traveltime derivatives.

The computation of the linear operator $\mathbf{G}_h$ requires the evaluation of the derivatives of traveltime and ray path at fixed reflector position (section 3.2.2) and changing surface point. For practical reasons, the system of equation (C.4) is solved starting from a fixed point at the surface and computing the raypath perturbations at the lower end of the ray. The traveltimes derivatives are equal in the two cases but the sign of the raypath derivatives must be changed.