A calculus for layered elastic media

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ABSTRACT

Group theory provides a framework for deriving the properties of the homogeneous elastic medium which is statically equivalent to a suite of anisotropic layers. Properties of a layer map reversibly to an element of a commutative group, where adding elements gives the group element for the homogeneous medium equivalent to the combination of two layers, and subtraction corresponds to the removing of layers. Fractures are also representable as group elements, allowing fractures and rocks to be manipulated together in a uniform manner. A well-defined subgroup structure helps sort out the special properties of symmetry classes.

INTRODUCTION

Previous work

The groundwork for this paper was laid in a previous report (Muir, 1987) which concluded

> *The Dix and Backus models—particularly in their extensions—are most simply manipulated, and their computer codes most simply maintained, when they are mapped over into their group representations.*

At much the same time, Schoenberg (Schoenberg & Douma, 1988) had modified and extended Backus' results in several ways.

1. Simplified the algebra by using a matrix formulation.
2. Extended the anisotropy model from Transverse Isotropic to Triclinic.
3. Included discussions on cracks and fractures.

This allowed Schoenberg and Muir to join forces and put together a rather general paper (Schoenberg & Muir, 1988) that cast these extended results in a group theoretic framework. In addition, they saw that there were useful connections between symmetry systems and subgroups. This paper, then, is a condensed version of the S&M paper which awaits publication.

Two models

The need to adopt a more complex model than the transverse isotropic one became clear at the Houston SEG Convention in 1986. Previous to this there had been two established points of view. On the one hand many workers in the oil industry saw transverse isotropy with a vertical symmetry axis as the principal useful model, with marine shales as the type rock. On the other hand Crampin and his followers, drawn generally from the global seismic and earthquake field, saw transverse isotropy with a horizontal axis as the important model, with vertical micro-cracks as the determining cause.

Reconciliation

Our thinking was changed by surface data presented by Alford, Lynn & Thomsen, and Willis &al; and VSP data presented by Johnson, and Becker & Perelberg. (All are reported in the SEG Extended Abstracts (1987).) These data indicated that both points of view are valid, and it is now clear that shear-wave splitting for vertically traveling waves is common in sedimentary basins, and that lack of splitting—the degenerate case—is the exception rather than the rule. The conclusion is that material in sedimentary basins will often require at least an orthorhombic model if elastic wave data is to be properly interpreted.

Fracture alignment

This splitting phenomenon is of special interest. If it results from aligned vertical cracks as proposed and discussed by Crampin (1985) and Crampin & Atkinson (1985), then not only is it useful to explorationists, but it also provides a new tool for development geologists and engineers in their reservoir modeling studies.

STATIC BEHAVIOR OF LAYERED MEDIA

Consider a medium made up of homogeneous layers with welded interfaces and let the x_3-axis be perpendicular to the layering. Assume that there are n different constituents. Each constituent has a relative thickness h_i, i=1,...,n (with $h_1 + ... + h_n = 1$), a density ρ_i, and an elastic modulus tensor, c_{pqrsi}, which relates stress, σ_{pqi}, with strain, ϵ_{rsi} according to the generalized Hooke's law $\sigma_{pqi} = c_{pqrsi}\,\epsilon_{rsi}$. In condensed notation, where $c_{pqrs} \rightarrow c_{jk}$ according to the usual convention, and omitting for now the layer indicator subscript i, the stress-strain relation in a layer may be written

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} \qquad (1),$$

where

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} .$$

The stiffness matrix is symmetric, so there can be at most 21 independent elastic constants. The elastic moduli for the equivalent medium can be expressed in terms of thickness-weighted averages of functions of the elastic moduli of the constituents. Since we are assuming static equilibrium, stress components normal to the layering and strain components tangential to the layering are the same in all layers, i.e.,

$$\begin{aligned} \sigma_{33i} &\equiv \sigma_{3i} = \sigma_3 , \quad \sigma_{23i} \equiv \sigma_{4i} = \sigma_4 , \quad \sigma_{13i} \equiv \sigma_{5i} = \sigma_5 , \qquad (2) \\ \epsilon_{11i} &\equiv \epsilon_{1i} = \epsilon_1 , \quad \epsilon_{22i} \equiv \epsilon_{2i} = \epsilon_2 , \quad 2\,\epsilon_{12i} \equiv \epsilon_{6i} = \epsilon_6 . \end{aligned}$$

The other stress and strain components, $\sigma_{11i} \equiv \sigma_{1i}$, $\sigma_{22i} \equiv \sigma_{2i}$, $\sigma_{12i} \equiv \sigma_{6i}$, $\epsilon_{33i} \equiv \epsilon_{3i}$, $2\,\epsilon_{23i} \equiv \epsilon_{4i}$, and $2\,\epsilon_{13i} \equiv \epsilon_{5i}$, may differ from layer to layer but will be constant within a layer. A concise way to find the effective moduli, even when constituent layers are anisotropic, is through a matrix formulation which follows.

The stress-strain relations (1) in the ith layer, taking the symmetry of the stiffness matrix c_{jki} into account, may be written

$$\begin{aligned} \sigma_{1i} &= c_{11i}\epsilon_1 + c_{12i}\epsilon_2 + c_{16i}\epsilon_6 + c_{13i}\epsilon_{3i} + c_{14i}\epsilon_{4i} + c_{15i}\epsilon_{5i} , \\ \sigma_{2i} &= c_{12i}\epsilon_1 + c_{22i}\epsilon_2 + c_{26i}\epsilon_6 + c_{23i}\epsilon_{3i} + c_{24i}\epsilon_{4i} + c_{25i}\epsilon_{5i} , \\ \sigma_{6i} &= c_{16i}\epsilon_1 + c_{26i}\epsilon_2 + c_{66i}\epsilon_6 + c_{36i}\epsilon_{3i} + c_{46i}\epsilon_{4i} + c_{56i}\epsilon_{5i} , \qquad (3) \end{aligned}$$

$$\begin{aligned} \sigma_3 &= c_{13i}\epsilon_1 + c_{23i}\epsilon_2 + c_{36i}\epsilon_6 + c_{33i}\epsilon_{3i} + c_{34i}\epsilon_{4i} + c_{35i}\epsilon_{5i} , \\ \sigma_4 &= c_{14i}\epsilon_1 + c_{24i}\epsilon_2 + c_{46i}\epsilon_6 + c_{34i}\epsilon_{3i} + c_{44i}\epsilon_{4i} + c_{45i}\epsilon_{5i} , \\ \sigma_5 &= c_{15i}\epsilon_1 + c_{25i}\epsilon_2 + c_{56i}\epsilon_6 + c_{35i}\epsilon_{3i} + c_{45i}\epsilon_{4i} + c_{55i}\epsilon_{5i} . \end{aligned}$$

Defining the following stress and strain vectors

$$\underline{S}_{1i} = \begin{bmatrix} \sigma_{1i} \\ \sigma_{2i} \\ \sigma_{6i} \end{bmatrix} , \quad \underline{S}_2 = \begin{bmatrix} \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} , \qquad \underline{E}_1 = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_6 \end{bmatrix} , \quad \underline{E}_{2i} = \begin{bmatrix} \epsilon_{3i} \\ \epsilon_{4i} \\ \epsilon_{5i} \end{bmatrix} \tag{4}$$

allows the stress-strain relations in the ith layer to be rewritten as

$$\underline{S}_{1i} = \mathbf{M}_i \, \underline{E}_1 + \mathbf{P}_i \, \underline{E}_{2i} \tag{5a}$$

and

$$\underline{S}_2 = \mathbf{P}_i^T \, \underline{E}_1 + \mathbf{N}_i \, \underline{E}_{2i} , \tag{5b}$$

where

$$\mathbf{M}_i = \begin{bmatrix} c_{11i} & c_{12i} & c_{16i} \\ c_{12i} & c_{22i} & c_{26i} \\ c_{16i} & c_{26i} & c_{66i} \end{bmatrix} , \quad \mathbf{N}_i = \begin{bmatrix} c_{33i} & c_{34i} & c_{35i} \\ c_{34i} & c_{44i} & c_{45i} \\ c_{35i} & c_{45i} & c_{55i} \end{bmatrix} , \quad \mathbf{P}_i = \begin{bmatrix} c_{13i} & c_{14i} & c_{15i} \\ c_{23i} & c_{24i} & c_{25i} \\ c_{36i} & c_{46i} & c_{56i} \end{bmatrix} . \tag{5c}$$

Superscript T denotes the matrix transpose. $\mathbf{M}_i$ and $\mathbf{N}_i$ are symmetric with 6+6=12 independent constants; $\mathbf{P}_i$, which is not symmetric, has 9 more so the total is 21.

Now denote the thickness weighted average $\sum_{i=1}^{n} h_i \, (\cdot_i)$ as $< \cdot >$. The moduli of the equivalent homogeneous medium are found by averaging. Averages of products of any of the 3×3 stiffness matrices $\mathbf{M}_i$, $\mathbf{N}_i$, or $\mathbf{P}_i$, which differ from layer to layer, with either $\underline{S}_{1i}$ or $\underline{E}_{2i}$, which differ from layer to layer in an unknown way, are indeterminate. It is necessary to solve Eqs.(5) for $\underline{S}_{1i}$ and $\underline{E}_{2i}$ before averaging.

To do this, premultiply (5b) by the matrix inverse $\mathbf{N}_i^{-1}$, solve for $\underline{E}_{2i}$, substitute the result into (5a) and collect terms. The resulting expressions are

$$\underline{S}_{1i} = \left[\mathbf{M}_i - \mathbf{P}_i \mathbf{N}_i^{-1} \mathbf{P}_i^T \right] \underline{E}_1 + \mathbf{P}_i \mathbf{N}_i^{-1} \, \underline{S}_2 \tag{6a}$$

and

$$\underline{E}_{2i} = -\mathbf{N}_i^{-1} \mathbf{P}_i^T \, \underline{E}_1 + \mathbf{N}_i^{-1} \, \underline{S}_2 , \tag{6b}$$

which when averaged give

$$< \underline{S}_1 > = \left[< \mathbf{M} > - < \mathbf{P} \mathbf{N}^{-1} \mathbf{P}^T > \right] \underline{E}_1 + < \mathbf{P} \mathbf{N}^{-1} > \underline{S}_2 \tag{6c}$$

and

$$< \underline{E}_2 > = - < \mathbf{N}^{-1} \mathbf{P}^T > \underline{E}_1 + < \mathbf{N}^{-1} > \underline{S}_2 . \tag{6d}$$

We have made use of the fact that $\underline{S}_2$ and $\underline{E}_1$ are constant to pull them outside the brackets. Note that since the matrices $\mathbf{M}_i$, $\mathbf{N}_i$, and $\mathbf{P}_i$ are known, all averages on the right hand side are well-defined.

To find the effective average stress-strain relationship for the stack of layers, we need to get (6c) and (6d) back into the form of Eqs. (5). To do this, we premultiply

(6d) by $< \mathbf{N}^{-1} >^{-1}$, solve for $\underline{S}_2$, substitute the result into (6c) and collect terms. This gives

$$< \underline{S}_1 > = \left[< \mathbf{M} > - < \mathbf{PN}^{-1}\mathbf{P}^T > + < \mathbf{PN}^{-1} >< \mathbf{N}^{-1} >^{-1}< \mathbf{N}^{-1}\mathbf{P}^T >\right] \underline{E}_1 + < \mathbf{PN}^{-1} >< \mathbf{N}^{-1} >^{-1} \underline{E}_2 \,, \qquad (7a)$$

$$\underline{S}_2 = < \mathbf{N}^{-1} >^{-1}< \mathbf{N}^{-1}\mathbf{P}^T > \underline{E}_1 + < \mathbf{N}^{-1} >^{-1} \underline{E}_2 \,. \qquad (7b)$$

Eqs.(7) are the correctly averaged stress-strain relations for a medium equivalent to the n interleaved constituent anisotropic media. The elastic moduli of the equivalent medium are obtained by comparison of Eqs.(7) with Eqs.(5), i.e.,

$$< \underline{S}_1 > = \mathbf{M} \; \underline{E}_1 + \mathbf{P} \; < \underline{E}_2 > \,, \qquad (8a)$$

$$\underline{S}_2 = \mathbf{P}^T \; \underline{E}_1 + \mathbf{N} \; < \underline{E}_2 > \,, \qquad (8b)$$

where

$$\mathbf{N} = < \mathbf{N}^{-1} >^{-1} \,, \qquad \mathbf{P} = < \mathbf{PN}^{-1} > \mathbf{N} \,, \qquad (8c)$$

$$\mathbf{M} = < \mathbf{M} > - < \mathbf{PN}^{-1}\mathbf{P}^T > + < \mathbf{PN}^{-1} > \mathbf{N} < \mathbf{N}^{-1}\mathbf{P}^T > \,.$$

The density of the equivalent medium is given by $< \rho >$.

GROUP THEORY

Geophysics is often viewed as a soft science in that it makes statements that are only more-or-less true about things that are only more-or-less clearly defined. One way around this mire of imprecision is to define a model and then discuss exact properties of the model. How well the model approximates the real world can then be separated out and left for discussion at another time and in another place. To some extent previous workers (including one of the authors) in this field have been negligent in introducing an approximating concept (long wavelength) at the beginning of discussions, which then taints any further arguments based on it. (Once an assumption has been made, it's usually impossible to see what would be changed if you tried to relax it again at some later point.) The attraction of group theory is that it enables us to make useful, precise statements about the behaviour of a precisely defined model: elastic layers in static equilibrium.

The elements of group theory as they apply to layered media were discussed in a previous paper (Muir, 1987), but it may be useful to recall the five properties of objects which form an Abelian group.

closure Layers add to form layers. To an external observer, the static behaviour of a set of heterogeneous layers is indistinguishable from that of the equivalent homogeneous layer.

association Too "obvious" to be deep?

commutation An external observer cannot distinguish how layers are ordered. A major analytical simplification that is true, curiously, both at the low and high frequency limits, but nowhere else inbetween.

identity element The "no layer". As a layer becomes increasingly thin, its influence disappears.

inversion A layer of negative thickness. The basis for subtraction.

COMBINING ANISOTROPIC LAYERS

A given anisotropic constituent, say the ith, is distributed in fine layers throughout the total thickness of a given region. Let the total thickness of all layers of the ith constituent in the region be H_i. Let this ith constituent medium have density ρ_i, and three 3×3 modulus matrices $\mathbf{N}_i$, $\mathbf{P}_i$ and $\mathbf{M}_i$. These five quantities, two scalars and three 3×3 matrices, can be thought of as physical model parameters. Two of the matrices, $\mathbf{N}$ and $\mathbf{M}$, are symmetric, and one of the matrices, $\mathbf{N}$, is invertible. For any such set of quantities, we can construct a new set of quantities $\mathbf{G}_i = [\, g_i(1),\, g_i(2),\, \mathbf{g}_i(3),\, \mathbf{g}_i(4),\, \mathbf{g}_i(5)\,]$ consisting of two scalars and three 3×3 matrices. The mapping is given by

$$\begin{bmatrix} H_i \\ H_i\,\rho_i \\ H_i\,\mathbf{N}_i^{-1} \\ H_i\,\mathbf{P}_i\,\mathbf{N}_i^{-1} \\ H_i\,[\mathbf{M}_i - \mathbf{P}_i\mathbf{N}_i^{-1}\mathbf{P}_i^T\,] \end{bmatrix} \rightarrow \begin{bmatrix} g_i(1) \\ g_i(2) \\ \mathbf{g}_i(3) \\ \mathbf{g}_i(4) \\ \mathbf{g}_i(5) \end{bmatrix} . \tag{9}$$

Note that $g_i(1)$ has dimensions of length, $g_i(2)$ has dimensions of length times density, $\mathbf{g}_i(3)$ has dimensions of length per stress, $\mathbf{g}_i(4)$ has dimensions of length, and $\mathbf{g}_i(5)$ has dimensions of length times stress. Also note that $g_i(1)$ is the total thickness of the ith constituent, $g_i(2)$ is the total mass of a column of unit area of the ith constituent and that $\mathbf{g}_i(j)$, $j = 3, 4, 5$ are H_i times coefficients that occur in Eqs.(6a) and (6b).

The set of all possible $\mathbf{G} = [\, g(1),\, g(2),\, \mathbf{g}(3),\, \mathbf{g}(4),\, \mathbf{g}(5)\,]$, consisting of two scalars and three 3×3 matrices with $\mathbf{g}(3)$ and $\mathbf{g}(5)$ symmetric, forms an Abelian group under addition. That is, the following group properties hold: 1) closure, 2) associativity, 3) commutativity, 4) the existence of a unique zero element $[0, 0, \mathbf{O}, \mathbf{O}, \mathbf{O}]$, and 5) for every group element $\mathbf{G}$, the existence of a unique inverse in the group, $-\mathbf{G}$. Call the Abelian group G. Eq.(9) is a mapping from a physical model to a group element and hereafter will be called the "group mapping".

For any $\mathbf{G}_i$ such that $\mathbf{g}_i(1) \neq 0$ and $\mathbf{g}_i(3)$ is invertible, we can return to the set of physical model parameters by the "inverse group mapping" from group element $\mathbf{G}_i$ to physical model. This inverse mapping is given by

$$\begin{bmatrix} g_i(1) \\ g_i(2)/g_i(1) \\ g_i(1)\mathbf{g}_i(3)^{-1} \\ \mathbf{g}_i(4)\mathbf{g}_i(3)^{-1} \\ [\mathbf{g}_i(5) + \mathbf{g}_i(4)\mathbf{g}_i(3)^{-1}\mathbf{g}_i(4)^T]/g_i(1) \end{bmatrix} \rightarrow \begin{bmatrix} H_i \\ \rho_i \\ \mathbf{N}_i \\ \mathbf{P}_i \\ \mathbf{M}_i \end{bmatrix} . \tag{10}$$

The group has been chosen in this way in order that the combination operation in the group be addition of the respective scalars and matrices, and that this should correspond to interleaving fine layers in the space of the physical model parameters. To see this consider the group element $\mathbf{G} = \sum_{i=1}^{n} \mathbf{G}_i$ so that

$$\begin{aligned} g(1) &= \sum_{i=1}^{n} g_i(1) = \sum_{i=1}^{n} H_i , \\ g(2) &= \sum_{i=1}^{n} g_i(2) = \sum_{i=1}^{n} H_i \rho_i , \\ \mathbf{g}(3) &= \sum_{i=1}^{n} \mathbf{g}_i(3) = \sum_{i=1}^{n} H_i \mathbf{N}_i^{-1} , \\ \mathbf{g}(4) &= \sum_{i=1}^{n} \mathbf{g}_i(4) = \sum_{i=1}^{n} H_i \mathbf{P}_i \mathbf{N}_i^{-1} , \\ \mathbf{g}(5) &= \sum_{i=1}^{n} \mathbf{g}_i(5) = \sum_{i=1}^{n} H_i [\mathbf{M}_i - \mathbf{P}_i \mathbf{N}_i^{-1} \mathbf{P}_i^T] , \end{aligned} \tag{11}$$

which on mapping back to physical model parameters according to (10) gives

$$\begin{aligned} H &= g(1) = \sum_{i=1}^{n} H_i , \\ \rho &= g(2)/g(1) = \sum_{i=1}^{n} H_i \rho_i / \sum_{i=1}^{n} H_i = < \rho > , \\ \mathbf{N} &= g(1)\mathbf{g}(3)^{-1} = \sum_{i=1}^{n} H_i \, (\sum_{i=1}^{n} H_i \mathbf{N}_i^{-1})^{-1} = < \mathbf{N}^{-1} >^{-1} , \\ \mathbf{P} &= \mathbf{g}(4)\mathbf{g}(3)^{-1} = \sum_{i=1}^{n} H_i \mathbf{P}_i \mathbf{N}_i^{-1} \, (\sum_{i=1}^{n} H_i \mathbf{N}_i^{-1})^{-1} = < \mathbf{P}\mathbf{N}^{-1} >< \mathbf{N}^{-1} >^{-1} , \\ \mathbf{M} &= [\mathbf{g}(5) + \mathbf{g}(4)\mathbf{g}(3)^{-1}\mathbf{g}(4)^T]/g(1) \end{aligned} \tag{12}$$

$$= \left[\sum_{i=1}^{n} H_i [\mathbf{M}_i - \mathbf{P}_i \mathbf{N}_i^{-1} \mathbf{P}_i^T] + \sum_{i=1}^{n} H_i \mathbf{P}_i \mathbf{N}_i^{-1} \, (\sum_{i=1}^{n} H_i \mathbf{N}_i^{-1})^{-1} \, (\sum_{i=1}^{n} H_i \mathbf{P}_i \mathbf{N}_i^{-1})^T \right] / \sum_{i=1}^{n} H_i$$

$$= <\mathbf{M}> - <\mathbf{PN}^{-1}\mathbf{P}^T> + <\mathbf{PN}^{-1}><\mathbf{N}^{-1}>^{-1}<\mathbf{N}^{-1}\mathbf{P}^T> ,$$

which correspond to the rules derived previously, Eqs.(8).

Adding group elements that correspond to stable elastic layers results in a new stable elastic layer. An elastic layer is stable if its 6×6 modulus matrix is positive definite. However, once we "subtract" layers, i.e., add the inverse group element that corresponds to the physical layer, $\mathbf{G} - \mathbf{G}_c = \mathbf{G}_b$, the remaining group element $\mathbf{G}_b$ need not correspond to a physical layer. That is, even if $g_b(1) \neq 0$ and $\mathbf{g}_b(3)$ is invertible so that operation (10) can be performed, the result could have non-positive thickness, i.e., $H_b \leq 0$; non-positive density i.e., $\rho_b \leq 0$; and/or be unstable, i.e. have a set of matrices $[\mathbf{N}_b, \mathbf{P}_b, \mathbf{M}_b]$ such that the overall 6×6 modulus matrix is not positive definite. However, if after subtraction $\mathbf{G}_b$ gives a stable medium b, it follows that $\mathbf{G}_b$ and $\mathbf{G}_c$ are a valid layer decomposition of the original medium corresponding to $\mathbf{G}$.

FRACTURES AS GROUP ELEMENTS

A set of parallel fractures can be treated as the limiting case of a set of layers whose elastic moduli approach zero as the thickness of the set of layers approaches zero (Schoenberg & Douma, 1988). Such a fracture system can be anisotropic in that it is not symmetric about an axis normal to the fractures. When the fracture set is symmetric about an axis normal to the fractures, Schoenberg & Douma have shown that it is indistinguishable from a dilute system of small, parallel, circular cracks as described by Hudson (1981). The two fracture system compliances, one normal and the other parallel to the fractures, are both proportional to crack density.

We next show how to transform a set of parallel fractures to a group element and thus how to fracture rocks. Let a particular constituent medium of a layered region of total thickness H be denoted with subscript f and have total thickness H_f, with $h_f \equiv H_f/H$. Further assume that its density and elastic moduli may be written $\rho_f = h_f\tilde{\rho}$, $\mathbf{N}_f = h_f\tilde{\mathbf{N}}$, $\mathbf{P}_f = h_f\tilde{\mathbf{P}}$ and $\mathbf{M}_f = h_f\tilde{\mathbf{M}}$. As $h_f \to 0$, this constituent approaches a soft medium with negligible inertia occupying a total region that becomes infinitesimally small, as the in-filling medium of a set of fractures. Hence the layers of this constituent become infinitely long fractures, each of which behaves as a linear slip interface. Across each slip interface, traction components σ_{3j} are continuous, as they are across any single layer in the static limit; but the components of displacement u_j need not be continuous, indicating that the strains in the fractures can become infinite as the moduli approach zero. The group mapping of physical properties of such a constituent that approximates a fracture set into a group element is

$$\begin{bmatrix} g(1) \\ g(2) \\ \mathbf{g}(3) \\ \mathbf{g}(4) \\ \mathbf{g}(5) \end{bmatrix} = \begin{bmatrix} h_f H \\ h_f^2 H \tilde{\rho} \\ H \tilde{\mathbf{N}}^{-1} \\ h_f H \tilde{\mathbf{P}} \tilde{\mathbf{N}}^{-1} \\ h_f^2 H [\tilde{\mathbf{M}} - \tilde{\mathbf{P}} \tilde{\mathbf{N}}^{-1} \tilde{\mathbf{P}}^T] \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ H \tilde{\mathbf{N}}^{-1} \\ \mathbf{O} \\ \mathbf{O} \end{bmatrix} \quad \textit{as } h_f \rightarrow 0. \quad (13)$$

Thus a fracture set made up of infinite, parallel slip joints can be characterized by at most a symmetric 3×3 matrix $\tilde{\mathbf{N}}^{-1}$ and depends on 6 scalars. This is the most general anisotropic fracture set. We define $\tilde{\mathbf{N}}^{-1} \equiv \mathbf{Z}$. $\mathbf{Z}$ is the excess compliance matrix from the fracture set, hereafter called the fracture set's compliance matrix.

The background medium is the medium before it is fractured. We denote its parameters with subscript b. Consider such a medium of thickness H into which is introduced a set of parallel fractures with compliance matrix $\mathbf{Z}$. The group element for this fractured medium is $\mathbf{G}_b$ combined with $\mathbf{G}_f$, yielding

$$\begin{bmatrix} H \\ H \rho_b \\ H(\mathbf{N}_b^{-1} + \mathbf{Z}) \\ H \mathbf{P}_b \mathbf{N}_b^{-1} \\ H(\mathbf{M}_b - \mathbf{P}_b \mathbf{N}_b^{-1} \mathbf{P}_b^T) \end{bmatrix} . \quad (14)$$

The inverse group mapping of (14), the set of parameters of the equivalent homogeneous medium, is by (10)

$$\begin{bmatrix} H \\ \rho_b \\ (\mathbf{N}_b^{-1} + \mathbf{Z})^{-1} \\ \mathbf{P}_b \mathbf{N}_b^{-1} (\mathbf{N}_b^{-1} + \mathbf{Z})^{-1} \\ \mathbf{M}_b - \mathbf{P}_b \mathbf{N}_b^{-1} \mathbf{P}_b^T + \mathbf{P}_b \mathbf{N}_b^{-1} (\mathbf{N}_b^{-1} + \mathbf{Z})^{-1} \mathbf{N}_b^{-1} \mathbf{P}_b^T \end{bmatrix} , \quad (15)$$

which corresponds to results of Schoenberg and Douma (1988).

Noting that $(\mathbf{N}_b^{-1} + \mathbf{Z})^{-1} \equiv \mathbf{N}_b (\mathbf{I} + \mathbf{Z}\mathbf{N}_b)^{-1}$, the changes in the moduli due to the fractures are

$$\begin{aligned} \Delta N &= \mathbf{N}_b [(\mathbf{I} + \mathbf{Z}\mathbf{N}_b)^{-1} - \mathbf{I}] , \\ \Delta P &= \mathbf{P}_b [(\mathbf{I} + \mathbf{Z}\mathbf{N}_b)^{-1} - \mathbf{I}] , \end{aligned} \quad (16)$$

and

$$\Delta M = \mathbf{P}_b [(\mathbf{I} + \mathbf{Z}\mathbf{N}_b)^{-1} - \mathbf{I}] \mathbf{N}_b^{-1} \mathbf{P}_b^T .$$

SYMMETRY SYSTEMS AS SUBGROUPS

Schoenberg & Muir (1988) present specifics on various symmetry systems and how they relate to a certain subgroup structure; we will not repeat that here. We shall discuss instead some more general ideas on subgroup structure and symmetries.

Although our group elements contain 29 numbers, the two scalars and three 3×3 matrices, symmetry conditions on two of the matrices reduce the maximum number of independent elements down to 23, corresponding to the thickness, density and possibly 21 independent elastic constants of the subject elastic medium. It follows that we can further abstract and simplify our group theory discussions by mapping each group element into an element of a new group of 23–vectors of reals. The combination operation is now vector addition, the inversion operation is vector negation, and the null vector is the unique identity element.

A particularly interesting class of subgroups is those that can be represented as a set of linear constraints on the vector element. Remembering that subgroups all share the identity element of the group, the null vector, we can see that the linear constraints must also be homogeneous. We conjecture that two more qualities are true of any subgroup that represents a symmetry.

1. *Some constraint equations will have one unity weight and the rest zero.*
2. *The balance of the constraint equations will have one unity weight, one minus unity weight, and the rest zero.*

In simple terms, elements of a vector will be unconstrained, equal to zero, or equal to another element. If this is true, then we have a simple way of categorizing anisotropy systems as they affect their static layered behaviour.

CONCLUSIONS

We have shown that each constituent layer of any anisotropy system can be mapped to an element of an Abelian group. This enables us to use elementary arithmetic to assemble and disassemble such systems,

$$\textbf{rock} + \textbf{rock} \leftrightarrow \textbf{rock}$$

and should simplify algorithmic design and maintainance. The decomposition property is reminiscent of Dix' velocity rule, and it should come as no surprise that this rule also has an Abelian group representation (Muir, 1987).

Since systems of parallel fractures may also be represented as group elements, rocks can be fractured and unfractured

$$\textbf{fracture system} + \textbf{rock} \leftrightarrow \textbf{fractured rock}$$

and this formalism extends to modeling rocks with more than one system of fractures. To model two sets of fractures, we rotate (back in model space) a rock with a single set of fractures to a coordinate system appropriate to the second fracture system, and then "add" the second set. As an example, two or more oblique sets of parallel vertical fractures, with or without horizontal bedding planes, give rise to a monoclinic medium with a vertical two-fold symmetry axis.

The representation of some symmetry systems, such as TI (transverse isotropic), as subgroups

$$\mathbf{TI} + \mathbf{TI} \rightarrow \mathbf{TI}$$

also much simplifies what otherwise is a quite cumbersome description of symmetry system hierarchy.

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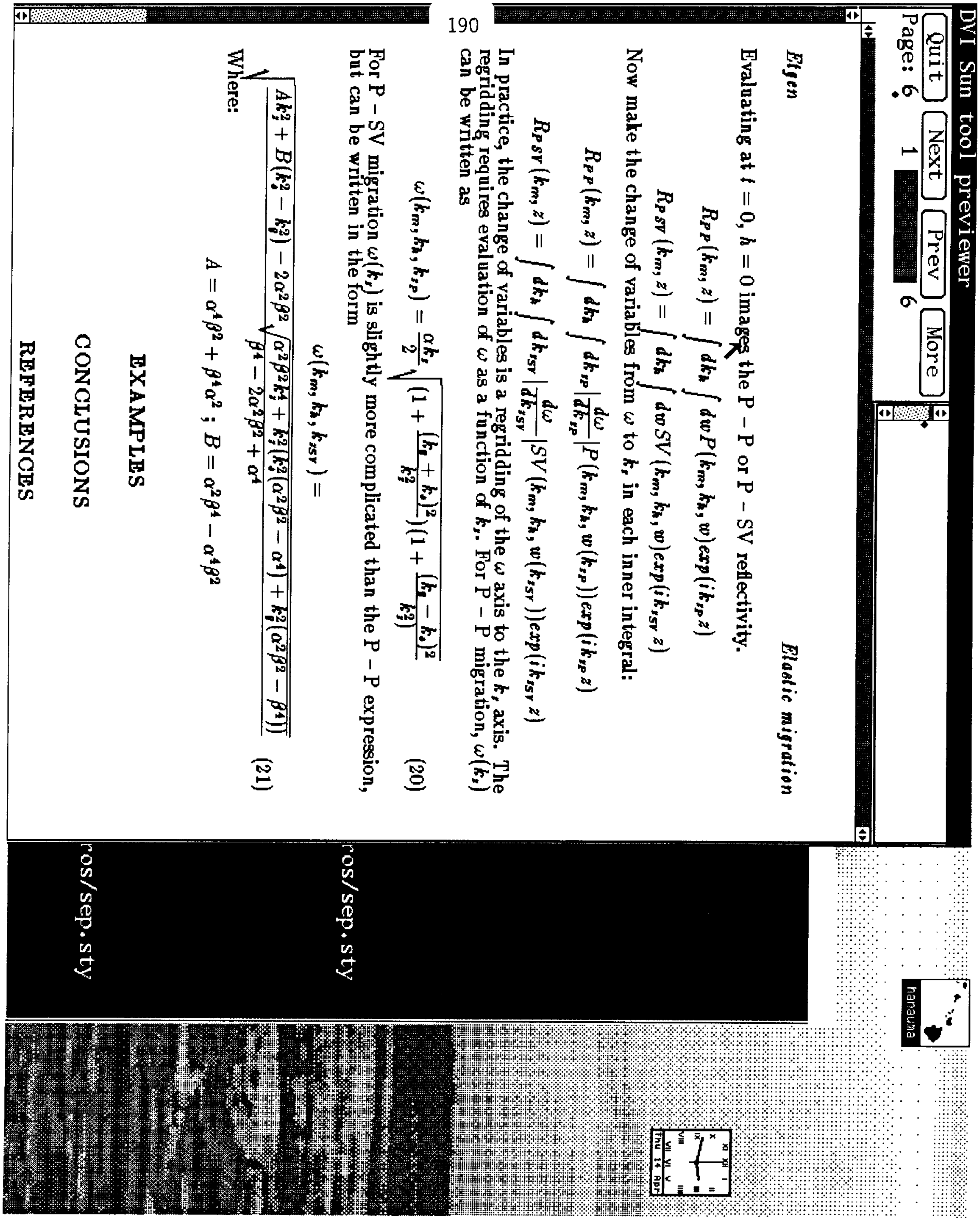

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