Eisner's reciprocity paradox and its resolution part I: the paradox

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Eisner's paradox reminds us to distinguish between energy and amplitude.

Ellipse definition

To construct an ellipse you start with two points on a line. The points are the *foci* of the ellipse. The line is the axis of the ellipse. Place a tack at each point. Loop a string from one tack to the other, leaving the string loose. Tighten the string by pushing it away from the tacks with the tip of a pencil. Moving the pencil around the tacks to all "tight-string" locations draws the ellipse. A ray emitted from one focus reflects from this ellipse and arrives at the other focus having traveled one string length, so all rays arrive at the other focus simultaneously.

EISNER'S COUNTER-EXAMPLE TO RECIPROCITY

Consider a perfectly reflecting ellipsoid with an impulsive source at one focus and a receiver at the other. Now, remove one end of the ellipsoid (as you might cut the end off a cucumber) by a cut through one focus perpendicular to the axis.

There are two experiments to consider: (A) source at the focus on the enclosed end, and (B) source at the focus on the cut-off end. These two experiments are shown together in Figure 1, with experiment A on the top and B on the bottom. In each case the source (marked by an S) emits rays evenly in all directions. Each ray can either escape through the hole in the cut-off end, or can reflect from the ellipse and arrive at the receiver (marked by an R).

A. Source at the interior focus (top)

When the source is at the interior focus, almost all the emitted energy reflects from the ellipse and then passes through the cut-end focus. Only a bit of the energy, much less than half, escapes through the hole. Of the rays shown in the figure, for example, only 2 escape, while 34 do not and thus arrive at the receiver.

B. Source at the cut-end focus (bottom)

When the source is at the cut-end focus *half* of the energy radiates to infinite space, never to

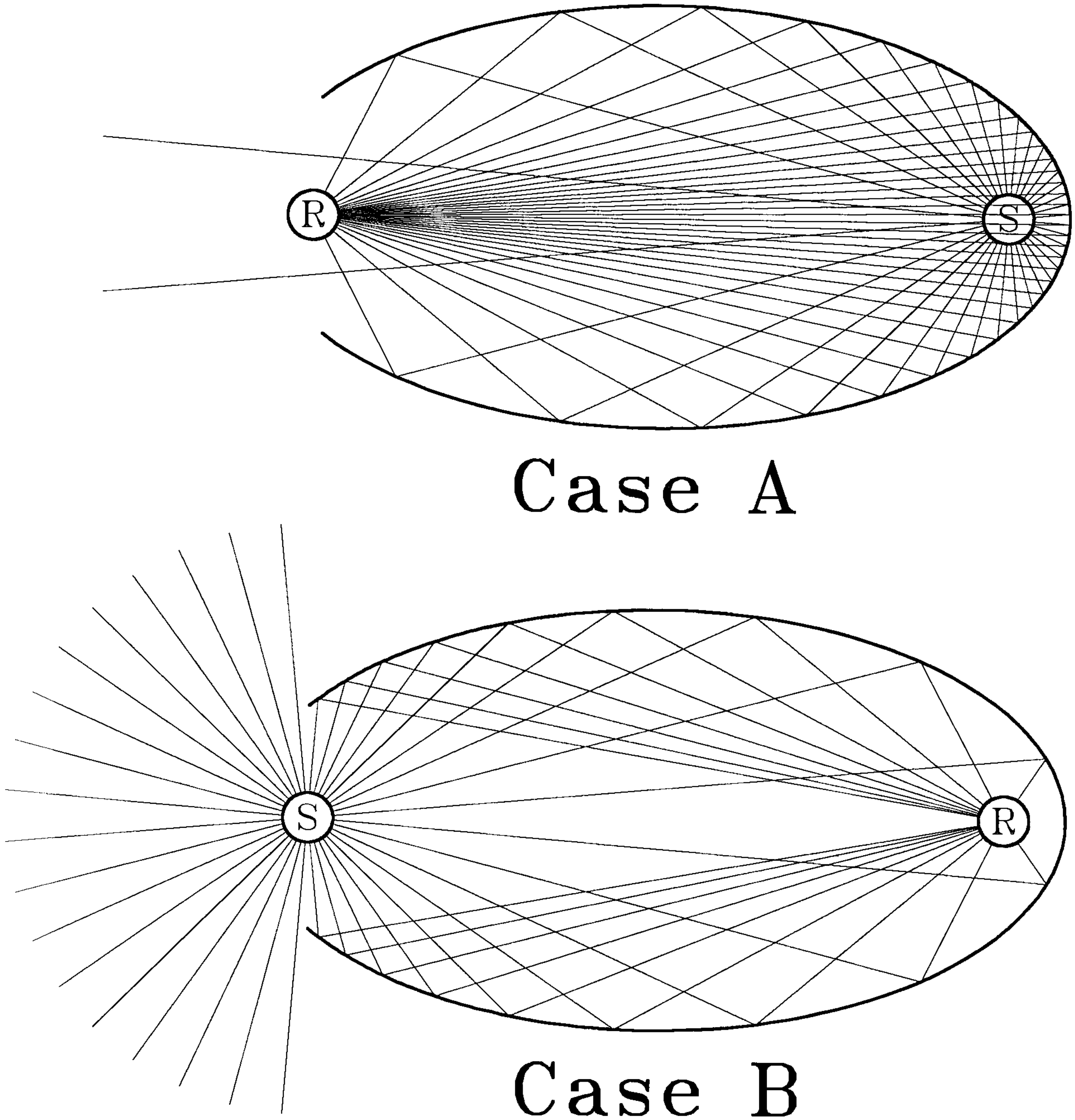

FIG. 1. Eisner's paradox. "S" is an isotropic source, and "R" is an isotropic receiver. As is obvious from the figure, nearly twice as much energy is recorded at the receiver in case A than there is in case B. Reciprocity demands that the complete seismogram recorded in each case be the same. Does reciprocity hold in this example?

return. The other half of the energy passes through the interior focus after one reflection from the ellipsoid.

Eisner's paradox

The reciprocal principle says that we should observe the same seismogram if we interchange an isotropic source with an isotropic receiver. The complete seismogram in case A must equal the complete seismogram in case B. The reciprocity principle seems to be violated here, because the primary reflection energy in experiment A is almost *twice* that in B.

This paradox was first presented at the 1980 SEG Friday workshop on the topic of "Seismic Reciprocity." Reciprocity is counter to intuition, and just when everyone in the audience was starting to believe in reciprocity, Elmer Eisner (like a magician) presented this example purporting to show that reciprocity contradicts energy conservation! He presented the paradox in a simple, direct way, much as above, with no mathematics. I (JFC) was there that day and heard much discussion of Elmer's talk but I do not recall anyone able to resolve his paradox, nor could I, so I urged Elmer to publish it. The paper appeared as a short note in Geophysics [1983]. Unlike the talk, the paper included a lot of mathematics — just the kind of distraction a clever magician might conjure up to distract an audience of theoreticians. Eisner states that half of the Geophysicists and Mathematicians he has presented this paradox to consider it to be a valid counterexample.

PREVIOUS EXPLANATIONS

In his original short note, Eisner gave several possible explanations, some of which were "false leads". One explanation was to simply note that according to geometrical optics, the amplitude at the focus in both cases is infinity, and twice infinity is still infinity, so reciprocity does not fail. Of course the amplitudes are not really infinite, because geometrical optics is only an approximation that fails when rays converge inside a region too small compared to the wavelength of the waves involved.

Another explanation was to note that the collapsing wavefront, while spherical, is not isotropic. If a pencil of energy with waves inside of a given amplitude impinges on a pressure detector at the focus, a wider pencil will have proportionately greater effect. In case A there are indeed more rays reaching the receiver than in B, but they are also much more concentrated in a single direction. This explanation gives a first hint as to how to resolve the paradox.

Dahlen and Odom's contribution

In 1984, F.A. Dahlen and R.I. Odom published a discussion ("Eisner's paradox resolved") in Geophysics, in which Eisner's hint is put on a mathematically precise footing. Did their discussion resolve the paradox? Perhaps. Anyway, you find a good clue following their equation (10):

> This (equation) does not present any conflict since there is no theorem guaranteeing reciprocity of received *energy*; what is guaranteed is reciprocity of the *amplitude* of the reflected arrivals (italics added).

Unfortunately, they throw you off track in the next sentence:

> The latter can prevail in the absence of the former since energy and amplitude are not simply related *at the foci,* where the signal is inherently a *diffraction phenomenon* due to the *failure of geometrical optics* (italics added).

Here is a better hint: Eisner's paradox has nothing to do with diffraction phenomena or the failure of geometrical optics at a focus (even though the paradox can be seen in the presence of diffraction or along with the occasional failure of geometrical optics). Eisner's paradox is a disguised form of a simple paradox that must occur to every new student of wave propagation in the first or second year of college physics. Now please study the diagrams and hints and contemplate the paradox **before** you turn forward to the resolution on page 107.