Wave field extrapolation by the linearly transformed wave equation operator

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ABSTRACT

Many approximations of different orders of the one-way wave equation have been suggested in seismic imaging or modeling. Of these approximations, the second-order approximation, usually called the 15 degree equation, is most commonly used in industry because of its high efficiency. However, all of these approximations have in common the constraints of not being able to handle the large angle events exactly.

Through a linear transformation of the wave equation, one can obtain, without approximation, the Linearly Transformed Wave Equation (LTWE) which exactly resembles in form a 15 degree equation. The solution to the LTWE is still a two-way wave solution. By imposing the upcoming (or downgoing) wave boundary condition, the LTWE can be applied to seismic imaging (or modeling). Implementing the LTWE with finite differencing algorithm gives an **one-hundred-and-eighty-degree,** or **all-dip,** finite-difference wave extrapolation operator, which solves the angle limitation problem in the conventional finite-difference methods.

INTRODUCTION

Most extrapolation methods used in seismic imaging (migration) and modeling, are based upon wave equation theory. In a two-dimensional elastic earth model, the wave equation is of the following form,

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0 , \tag{1}$$

where P is wavefield, x horizontal coordinate, z depth, t time and v velocity. The extrapolation of wavefield along the positive z direction (migration) , or along the negative z direction (modeling), requires equation (1) to be changed into the one-way wave equation,

$$\left\{ \frac{\partial}{\partial z} \pm \frac{1}{v} \frac{\partial}{\partial t} \left[1 - v^2 \frac{\frac{\partial^2}{\partial x^2}}{\frac{\partial^2}{\partial t^2}} \right]^{1/2} \right\} P = 0 , \qquad (2)$$

where the positive sign is used in modeling, the negative in migration.

When solving the above second order differential equation, one must have not only the boundary, or initial, conditions of wavefield itself, but also the boundary, or initial, conditions of the first derivatives of wave filed. Unfortunately, these first derivatives are presently not recorded in routine seismic surveys (Ma, 1982). Fourier domain extrapolation algorithms, such Phase-shift and Stolt, eliminate the derivative requirement by transforming the derivatives in the time-space domain into the polynomials in the Fourier domain. With approximation, time-space (or frequency-space) domain extrapolation algorithms, such as finite-differences, avoid the first derivative requirement by taking finite terms in Taylor's series expansion (Claerbout, 1976), or using certain order of Muir's recursive expansion (Claerbout, 1982), of the square-root operator in the one-way wave equation, and by imposing certain lateral boundary conditions, such as absorbing boundary condition, as well as natural boundary conditions, such as the upcoming wave boundary conditions. The finite-difference methods, especially in time-space domain, are generally more efficient than the other methods, because their computational costs are lower. Therefore, in routine seismic data processing, most industries use finite-difference migration (FDM), especially time domain FDM.

Finite difference wavefield extrapolation in either time-spatial coordinates, (t ,z ,x), or frequency-spatial coordinates, (ω,z ,x), is based on different approximating equations obtained by either expanding the square root operator in equation (2), or by transforming wavefield coordinate system, such as retarded coordinate transform or moving coordinate transform. All the approximating equations replace the total operator in equation (2) with finite terms of differentiation operators. The residual error in the operator approximation is proportional to a certain power of $\sin(\theta)$, where θ is angle of wave propagation (angle measured from vertical axis). Therefore, algorithms based on these approximating equations are not accurate when applied to events with large propagating angles. Taking more terms in the square-root expansion will increase the accuracy of extrapolating the events with large propagating angles. However, the computational cost increases significantly when higher order approximating equations are used.

The approximation most commonly used in industry is the second-order approximation of equation (2), usually called the 15 degree equation; when only the first two terms in the square-root expansion are taken and the retarded coordinate transform is applied: $t' = t \pm z/v$, (+ for migration, – for modeling), $z' = z$ and $x' = x$, equation (2) becomes:

$$\left\{ \frac{\partial^2}{\partial z' \, \partial t'} \pm \frac{v}{2} \frac{\partial^2}{\partial x'^{\,2}} \right\} P = 0 , \tag{3}$$

where – is used for modeling and + for migration. It is easy to implement equation (3) in (t', z', x') space using finite-difference, because the second-order t' and z' derivatives of P are not involved.

A nonsingular linear transformation of the coordinates can transform the last two terms in equation (1) into a single cross-derivative term; i.e.,

$$\left\{ \frac{\partial^2}{\partial x'^{\,2}} + a \frac{\partial^2}{\partial z' \, \partial t'} \right\} = 0 , \tag{4}$$

where a is a coefficient that depends on the transform and v. This equation, without approximation, is valid for all events with any propagating angles.

This paper presents the linear transformation that transforms the full wave equation into the Linearly Transformed Wave Equation (LTWE), which exactly resembles in form a conventional 15 degree equation. A finite-difference algorithm of LTWE and some computational examples are also discussed in the paper.

REVIEW OF CONVENTIONAL FINITE DIFFERENCE MIGRATION

There are several ways to get an approximating equation for finite-difference migration. We rewrite equation (2) as it is used for migration,

$$\left\{ \frac{\partial}{\partial z} - \frac{1}{v} \frac{\partial}{\partial t} \left[1 - v^2 \frac{\dfrac{\partial^2}{\partial x^2}}{\dfrac{\partial^2}{\partial t^2}} \right]^{1/2} \right\} P = 0 . \tag{5}$$

The square root must be expanded and approximated by a finite number of terms, if a finite-difference algorithm is to be used. Muir's iterative expansion of the square root, $R = \sqrt{1 - X^2}$, is of the following form,

$$R_{n+1} = 1 - \frac{X^2}{1 + R_n} , \tag{6}$$

where $X^2 = v^2 \frac{\partial^2/\partial x^2}{\partial^2/\partial t^2}$, $R_0 = 1$.

The Taylor's series expansion of the square-root operator is,

$$R = \frac{1}{v}\frac{\partial}{\partial t}\left[1 - \frac{v^2}{2} \frac{\frac{\partial^2}{\partial x^2}}{\frac{\partial^2}{\partial t^2}} + \textit{higher order terms} \cdots \right] . \tag{7}$$

Dropping higher-order terms in the square root expansion, we get the second-order migration equation,

$$\left\{ \frac{\partial}{\partial z} - \frac{1}{v}\frac{\partial}{\partial t}\left[1 - \frac{v^2}{2} \frac{\frac{\partial^2}{\partial x^2}}{\frac{\partial^2}{\partial t^2}} \right] \right\} P = 0 . \tag{8}$$

Using the retarded coordinate transform: $t' = t + z/v$, $z' = z$ and $x' = x$, equation (8) is transformed to the commonly used 15 degree migration equation:

$$\left\{ \frac{\partial^2}{\partial z' \, \partial t'} + \frac{v}{2} \frac{\partial^2}{\partial x'^2} \right\} P = 0 . \tag{9}$$

Another way of getting the approximating equation for migration is to use a retarded coordinate transformation, (Stolt, 1978)

$$\begin{cases} x' = x \\ z' = z \\ t' = t + z/v \end{cases} . \tag{10}$$

Plugging transform (10) into wave equation (1), we have

$$\left\{ \frac{\partial^2}{\partial x'^2} + \frac{2}{v} \frac{\partial^2}{\partial z' \, \partial t'} + \frac{\partial^2}{\partial z'^2} \right\} P = 0 . \tag{11}$$

Dropping out the second z' derivative term gives the same approximating equation as equation (9).

These approximating equations have in common the dip restriction, because the dropped higher-order derivative terms are important in characterizing the large angle propagating events. One can obtain higher-order equations either by taking more terms in expansion, or by taking higher-order derivatives of z' in the retarded coordinate wave equation and then dropping out the terms of higher derivatives of z' . The higher the order, the better the approximation in the migration of dipping reflections. However, when the dip of the reflector exceeds 45 degree, the higher-order equations are difficult to realize and the increased cost for doing so is very significant, though Ma's splitting methods can be used (Ma, 1982, and Jacobs, 1982).

We look for a proper transformation such that the resulting transformed equation is an exact representation of the original wave equation and is also easily realized by finite differencing methods.

THE LINEAR TRANSFORMATION

For a general quadratic form,

$$F(X) = \sum_{i=1,j=1}^{n} a_{ij} x_i x_j = X^{-1}AX, \tag{12}$$

there generally exists a nonsingular linear transform,

$$Y = L X , \tag{13}$$

such that

$$F(X) = Y^{-1}LAL^{-1}Y = Y^{-1}BY = \sum_{i=1}^{n} b_i y_i^2 = G(Y), \tag{14}$$

where G(Y) is a standard quadratic form.

Because L is nonsingular, we can transform G(Y) back to F(X) by the inverse transform, $X = L^{-1}Y$.

Now, we are going to apply these concepts to the transformation of the wave equation. The two-dimensional full wave equation is,

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0 . \tag{15}$$

Let $y_1 = \partial/\partial x$, $y_2 = \partial/\partial z$, $y_3 = \partial/\partial t$, $b_1 = b_2 = 1$, $b_3 = -1/v^2$; the left side of equation (15) becomes a standard quadratic form,

$$G(Y) = \sum_{i=1}^{3} b_i y_i^2. \tag{16}$$

We can find a nonsingular linear transform, $X = LY$, to eliminate y_2^2 and y_3^2, and get a cross term $x_2 x_3$, where $x_1 = \partial/\partial x'$, $x_2 = \partial/\partial z'$, $x_3 = \partial/\partial t'$. The resulting transformed equation is of the following form,

$$\frac{\partial^2 P}{\partial x'^2} + a \frac{\partial^2 P}{\partial z' \partial t'} = 0 , \tag{17}$$

where a is a coefficient determined by the transform coefficients and v.

The linear transformation is of the following form,

$$\begin{cases} x' = x \\ z' = c_1 z + d_1 t \\ t' = c_2 z + d_2 t \end{cases} . \tag{18}$$

Using the chain rule for derivatives, we get

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x'} \\ \dfrac{\partial}{\partial z} = c_1 \dfrac{\partial}{\partial z'} + c_2 \dfrac{\partial}{\partial t'} \\ \dfrac{\partial}{\partial t} = d_1 \dfrac{\partial}{\partial z'} + d_2 \dfrac{\partial}{\partial t'} \end{cases} . \tag{19}$$

Taking the second derivatives (square of first derivative) and substituting them into the wave equation (15), we can determine the coefficients used in the linear transformation by equating the coefficients in the left sides of equations (15) and (17),

$$\begin{cases} c_1^2 - d_1^2/v^2 = 0 \\ c_2^2 - d_2^2/v^2 = 0 \\ 2c_1 c_2 - 2d_1 d_2/v^2 = a \end{cases} . \tag{20}$$

Equation (20) is a general constraint for the transform coefficients. When $c_1 = 1/\sqrt{2}$, $d_1 = -v/\sqrt{2}$, $c_2 = 1/(v\sqrt{2})$, and $d_2 = 1/\sqrt{2}$, a special case of the linear transformations (unitary when regarding vt and vt' as variables) takes the following form,

$$\begin{cases} x' = x \\ z' = \dfrac{1}{\sqrt{2}} \left\{ z - vt \right\} \\ t' = \dfrac{1}{\sqrt{2}} \left\{ z/v + t \right\} \end{cases} , \tag{21}$$

the inverse transformation is,

$$\begin{cases} x = x' \\ z = \dfrac{1}{\sqrt{2}} \left\{ z' + vt' \right\} \\ t = \dfrac{1}{\sqrt{2}} \left\{ t' - z'/v \right\} \end{cases} , \tag{22}$$

The resulting equation is, then

$$\frac{\partial^2 P}{\partial x'^{\,2}} + \frac{2}{v}\frac{\partial^2 P}{\partial z'\,\partial t'} = 0\ , \tag{23}$$

Equation (23) has been exactly transformed from full wave equation without any terms having been dropped; it is one format of the LTWE defined in equation (17). Therefore, there is no dip constraint upon using it for wavefield extrapolation. The LTWE is, in format, the same as the conventional 15 degree type equation and is easy to code into computer.

Another and easier way of deriving the LTWE can be obtained in the Fourier domain. The dispersion relation of wave equation is, by 3-D Fourier transforming equation (2),

$$k_x^{\,2} + k_z^{\,2} - \frac{\omega^2}{v^2} = 0\ , \tag{24}$$

or,

$$k_x^{\,2} + (k_z + \frac{\omega}{v})(k_z - \frac{\omega}{v}) = 0\ . \tag{25}$$

Let

$$\begin{cases} k_{z_1} = k_z + \dfrac{\omega}{v} \\ k_{z_2} = k_z - \dfrac{\omega}{v} \end{cases}\ , \tag{26}$$

we get the dispersion relation of wave equation in (k_{z_1}, k_{z_2}, k_x),

$$k_x + k_{z_1} k_{z_2} = 0\ . \tag{27}$$

Transforming equation (27) back to (z_1, z_2, x), we can obtain one form of the LTWE in equation (17) with $a = 1$.

FINITE DIFFERENCING IMPLEMENTATION OF THE LTWE

The two solutions of the LTWE and the upcoming wave boundary condition

The LTWE is linearly transformed from the full-way equation. Therefore, it is also a two-way wave equation: it handles both the upcoming waves and the downgoing waves. In order to do migration, we must impose a certain wavefield condition to separate the upcoming waves from the downgoing waves as much as possible.

Let's study the cross-derivative terms in both the LTWE and the 15^0 equation. When the wavefield is independent of the horizontal axis as in the one-dimensional wave propagation, both the LTWE and the 15 degree equations take the following form,

$$\frac{\partial^2 P}{\partial \xi \partial \eta} = 0 . \tag{28}$$

The solution of equation (28) is, called D'Alembert's solution,

$$P = \Phi(\xi) + \Psi(\eta) . \tag{29}$$

For the LTWE case, $\xi = t + z/v$ and $\eta = t - z/v$. The solution of equation (28) has both the upcoming wave solution $\Phi(\xi) = \Phi(t + z/v)$ and the downgoing wave solution $\Psi(\eta) = \Psi(t - z/v)$. Let

$$P \Big|_{t+z/v > T_{\max}} = 0 , \tag{30}$$

Then, the maximum depth where the wavefield can be recorded is $z_{\max} = vT_{\max}$. The wave generated at this depth at $t = 0$ can only travel upwards (z must decrease) as time increases, because of condition (30). Therefore, condition (30) is called *the upcoming wave boundary condition.* However, the other energy generated at any depth z_0 above $z_{\max}$ can travel both upwards and downwards if $\Delta t + \Delta z / v + z_0$ does not exceeds $T_{\max}$ ($\Delta z > 0$ when traveling downwards, $\Delta z < 0$ when traveling downwards).

For the 15 degree migration equation case, the variables in equation (29) correspond to: $\xi = t + z/v$ and $\eta = z$. The solution is,

$$P = \Phi(t + z/v) + \Psi(z) , \tag{31}$$

which is the sum of an upcoming wave solution $\Phi(t + z/v)$ and a time-independent function $\Psi(z)$. Therefore, the 15 degree migration equation eliminates the downgoing waves completely by dropping off the higher-order terms in the expansion of the square-root operator.

In two-dimensional wave propagation, P is also a function of the variable x. The above reasoning for decomposing P into either an upcoming-wave solution or a downgoing-wave solution will be still valid, when events are propagating with small angles (as in a flat-layered medium) and when no reverberations are taken into account (as in migrating primary reflections).

It will be shown in the later sections of this paper that imposing the upcoming wave boundary condition with the LTWE, we will be able to extrapolate the reflection data downwards to the reflectors without the requirement of knowing the first derivatives of the wavefields.

Finite-difference implementation of the LTWE migration

One of the important decisions in finite-difference techniques is the choice of boundary conditions. In performing seismic migration, we start with surface *data* $P(t,z=0,x)$ and migrate to get an *image* $P(t=0,z,x)$, under *the upcoming wave boundary condition,* $P(t,z,x)\Big|_{t+z/v>T_{\max}}=0$.

In the new LTWE coordinate system, (t',z',x'), these conditions correspond to the following: *data* $P(t'=t/\sqrt{2},z'=-vt/\sqrt{2},x'=x)$; *image* $P(t'=z/(v\sqrt{2}),z'=z/\sqrt{2},x'=x)$; and, *the upcoming wave boundary condition,* $P(t'>T_{\max}/\sqrt{2},z',x')=0$. The finite-differencing grids of the LTWE are shown in Figure 1. The surface data lies along the line $z'=-vt'$, while the image along $z'=vt'$. With the upcoming wave boundary condition, the *data* can be extrapolated from the t axis to the z axis to get the *image*. Figure 1 is also valid for using the LTWE to do modeling. It turns out that the casuality conditions (with respect both to t and z) of the wavefield should be used instead of the upcoming wave boundary condition, when the LTWE finite-difference algorithm is to be used in modeling. The LTWE with the casuality conditions will give a stable finite-difference solution to the modeling problem.

The physical interpretation of the LTWE is that the wave propagation can be characterized by the LTWE if we rotate the coordinate (vt,z) by 45 degree. The rotation does not change the orthogonality of the coordinate system, while the retarded coordinate transform does change the orthogonality. Under the rotated coordinate (vt',z'), the wavefield solution can be represented by the sum of two functions, with one function depends on (z',x') only and the other depends on (t',x') only.

The geometrical interpretation of the LTWE is that it transforms a five finite differencing star pattern in (t,z) into a four finite differencing star pattern in $(t'\ z')$ by the 45 degree rotation. The four star pattern does not require the first derivatives of wave field to be known, while the five star pattern does. The relation between these two finite-difference star patterns are shown in Figure 2.

The finite-difference stars used in the LTWE are the same as in the conventional 15 degree equation migration (Claerbout, 1976), as shown in Figure 3. The finite differencing in x' is chosen to be of the implicit form, because of the stability consideration (Claerbout, 1982). The finite-difference star in the (t',z') plane can move either upward or leftward. The resulting finite differencing representation of equation (23), in the (t',z') plane is,

$$\left(I+\alpha T\right)P_{t',z'+\Delta z'}=\left(I-\alpha T\right)\left\{P_{t',z'}+P_{t'+\Delta t',z'+\Delta z'}\right\}-\left(I+\alpha T\right)P_{t'+\Delta t',z'} \quad (32)$$

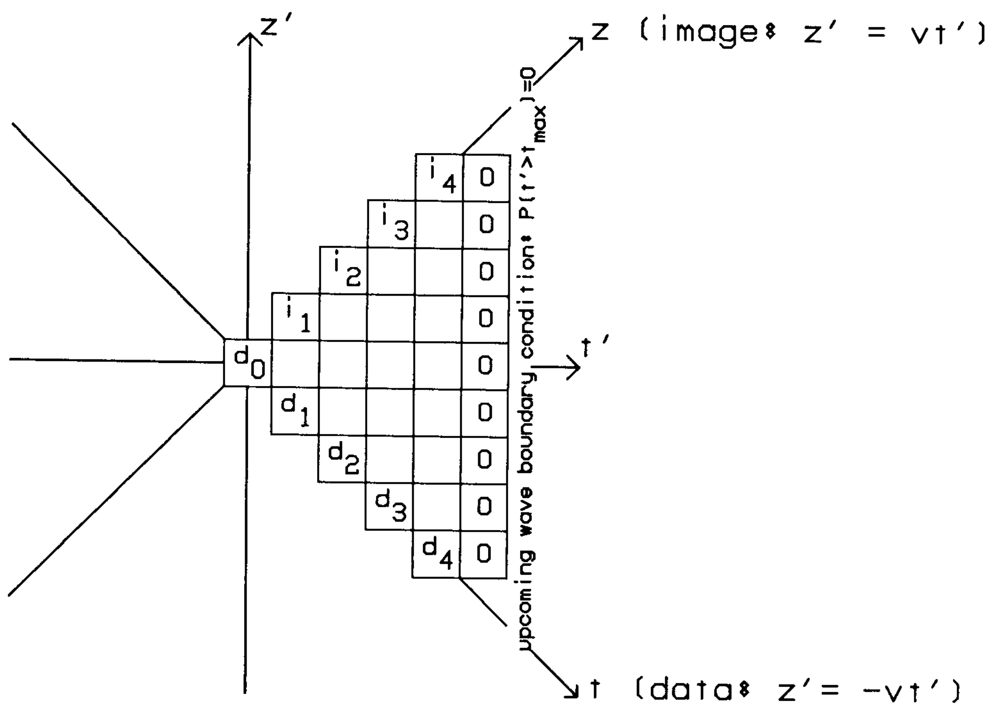

FIG. 1. Finite difference in (t', z', x'). The length of data is assumed to be 5 in the figure. *Data* $P(x, z=0, t)$ is placed at d_0, d_1, ... , d_4. *Image* $P(x, z, t=0)$ is obtained at d_0, i_1, ..., i_4.

where $\alpha = v \Delta t' \Delta z' / 8 \Delta x'^{2}$. The differencing operators, $T = (-1,2,-1)$ and $I = (0,1,0)$, are applied along the x' axis. The elements of the unknown vector in the left-hand side of equation (32) are given implicitly by the elements of the three known vectors in the right-hand side of equation (32). For each step of moving finite-differencing star in (t', z'), a tridiagonal-matrix-solver routine is called to get the wavefield along the whole x' axis. It turns out that the computer algorithm using equation (32) is unconditionally stable (Mitchell, 1980). Finite-differencing stars in (t', z') and (x', z') are shown in Figure 3.

A RATFOR (RATional FORtran) subroutine of doing LTWE migration is given in the appendix of this paper.

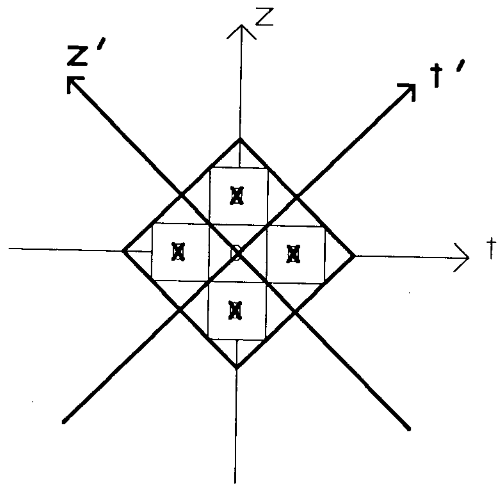

FIG. 2. Finite difference star patterns in (t,z) and in (t',z'). A five differencing star pattern representing $(\partial^2/\partial z^2 - v^{-2}\partial^2/\partial t^2)$ in (t,z) is transformed to a four differencing star pattern representing $a\,\partial^2/\partial z'\,\partial t'$ in (t',z') by a 45 degree rotation of the coordinate axes. The five star pattern is drawn in the light lines, while the four star pattern in the heavy lines. The original elements upon which the five star pattern is applied are shown by the symbol O, while the elements upon which the four star pattern is applied are shown by the symbol X.

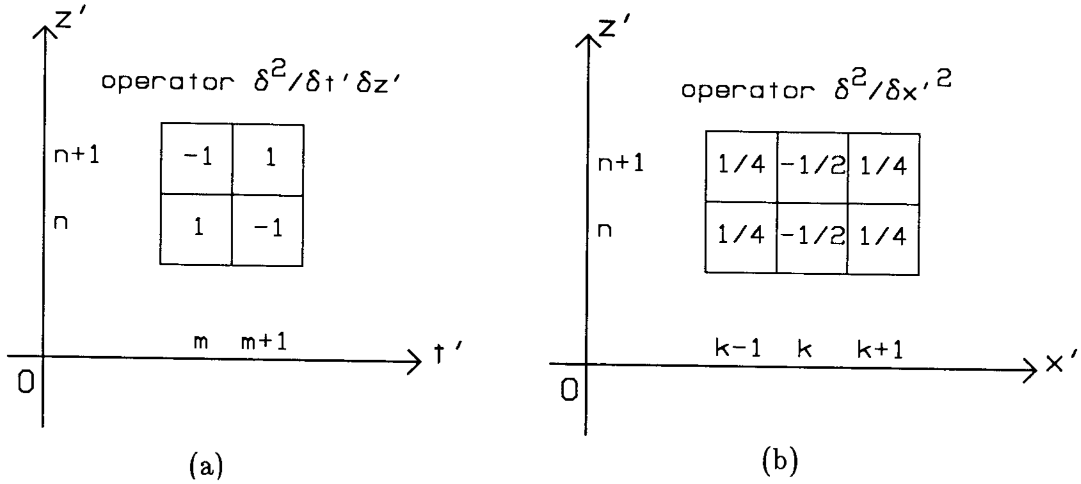

FIG. 3. (a) Finite-differencing star of $\delta^2/\delta t'\,\delta z'$ in (t',z'). (b) Finite-differencing star of $\delta^2/\delta x'^2$ in (x',z'). The operator $\delta^2/\delta x'^2$ has been averaged over the four elements in (t',z') plane. The finite-differencing star of $\delta^2/\delta x'^2$ in (x',t') is the same as it is in (x',z').

Examples of using the LTWE migration on synthetic data

Figure 4(a) shows the impulse response of the LTWE operator. The semicircle response is well preserved, which shows that the LTWE is valid for events with any propagating angle. For comparison, Figure 4(b) shows an impulse response of a 15 degree operator. The semicircle is distorted by the 15 degree operator, because the large angle events corresponding to the upper part of the semicircle have been distorted and eliminated by the dip-filter effect of the 15 degree operator. It is clear that the LTWE operator is much better than the lower-order 15 degree one.

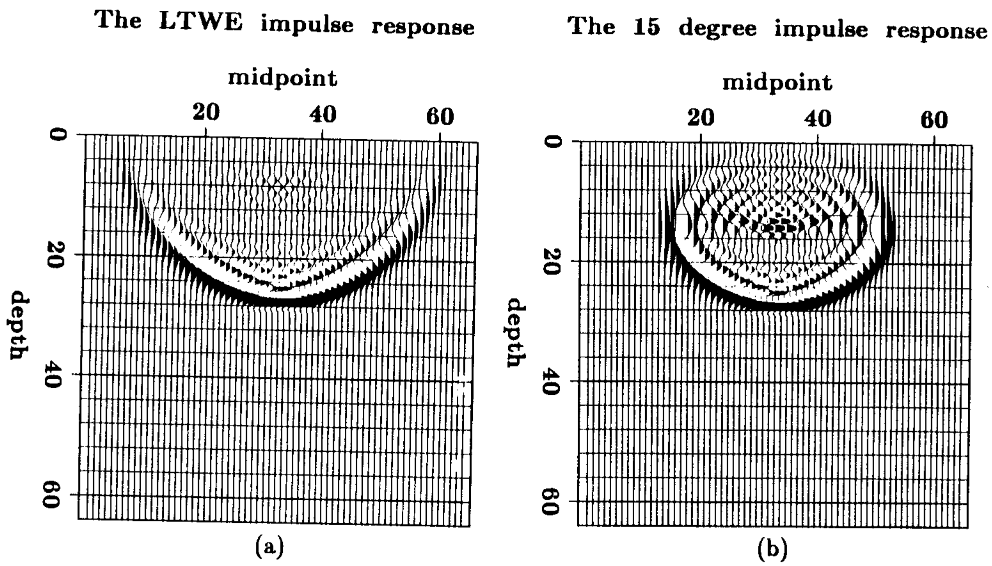

FIG. 4. (a) Impulse response of the LTWE operator. (b) Impulse response of the 15 degree operator. The plotting parameters of both figures are the same. The velocity of the model is constant. The finite-differencing dispersion effect shown on the upper wiggling part of (b) is greatly reduced on (a).

Figure 5 shows a comparison of the results of applying different migration operators to the same synthetic data. The original model is composed of 5 segments of lines having 5 different dipping angles. The data is generated by phase-shift modeling. The four different migration operators are: (1) 15 degree in time-space domain; (2) 15 degree in frequency-space domain; (3) 45 degree in frequency-space domain; and (4) the LTWE.

The small-angle reflectors are imaged well by all of the four methods. However, the reflectors with large angles, especially the steepest one, are well imaged only by the LTWE operator. The comparison between the LTWE and the conventional 15 degree and 45 degree migration operators shows that the LTWE one is promising.

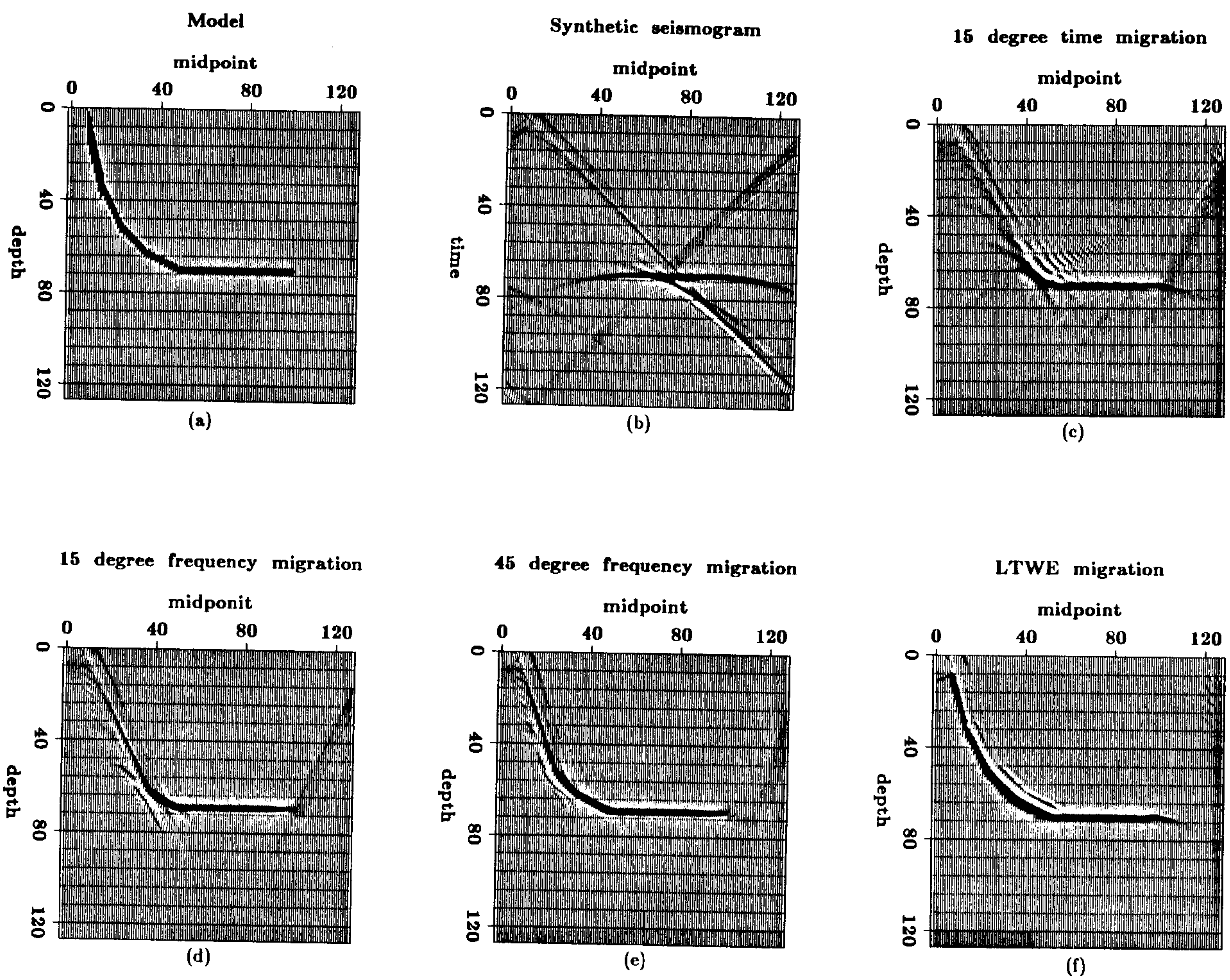

FIG. 5. Comparison between 4 different migration operators. (a) The 5-segment dipping-bed model. The slopes of the five segments are: 0, 0.5, 1, 2 and 4, respectively. The steepest one has the angle about 76 degree. Velocity is constant in the model. (b) The synthetic seismogram generated with phase-shift modeling. (c) Migration with 15 degree operator in time-space domain. (d) Migration with 15 degree operator in frequency-space domain. (e) Migration with 45 degree operator in frequency-space domain. (f) Migration with the LTWE operator.

The LTWE in velocity varying media

So far, we have not taken into account the variation of v. When the velocity is varying, there will be an error term in the LTWE; this term also exists in the all approximating one-way wave equations. It turns out that the error caused by the varying velocity is proportional to the velocity gradient divided by the velocity .

In order to check the error caused by varying velocity, let's derive the LTWE in another way. Starting with the full wave equation (1),

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} = 0 , \tag{1}$$

and letting $dz = v(\tau) d\tau$, where τ is the so-called two-way vertical time, we have,

$$\frac{\partial^2}{\partial z^2} = \left\{ \frac{1}{v(\tau)} \frac{\partial}{\partial \tau} \right\} \left\{ \frac{1}{v(\tau)} \frac{\partial}{\partial \tau} \right\} = \frac{1}{v^2(\tau)} \frac{\partial^2}{\partial \tau^2} - \frac{1}{v^3(\tau)} \frac{\partial v(\tau)}{\partial \tau} \frac{\partial}{\partial \tau} . \tag{33}$$

The wave equation in (t,τ,x) is,

$$\frac{\partial^2 P}{\partial x^2} + \frac{1}{v^2(\tau)} \frac{\partial^2 P}{\partial \tau^2} - \frac{1}{v^2(\tau)} \frac{\partial^2 P}{\partial t^2} = \frac{1}{v^3(\tau)} \frac{\partial v(\tau)}{\partial \tau} \frac{\partial P}{\partial \tau} . \tag{34}$$

Now take the **velocity independent** linear transform:

$$\begin{cases} t_1 = \dfrac{1}{\sqrt{2}} \left\{ \tau + t \right\} \\ t_2 = \dfrac{1}{\sqrt{2}} \left\{ \tau - t \right\} \end{cases} . \tag{35}$$

Then, the left-hand side of equation (34) is transformed to the left-hand side of the LTWE operator (17). Because transform (35) is velocity independent, the transform, (35), itself will introduce nor error. Therefore, the right-hand side of equation (34) is the error term in the LTWE (17), when velocity is varying. The error term is,

$$error = \frac{1}{v^3(\tau)} \frac{\partial v(\tau)}{\partial \tau} \frac{\partial P}{\partial \tau} = \frac{1}{v} \frac{\partial v}{\partial z} \frac{\partial P}{\partial z} . \tag{36}$$

Hence, the error term is proportional to velocity gradient divided by velocity, times the wavefield gradient. This term can be easily coded into the LTWE operator, and will be very significant in the modeling of both reflections and reverberations. Because the velocity usually changes smoothly, the velocity-error term is actually negligible in the LTWE migration. The velocity error term must be neglected if one is interested in migrating the primary reflections only.

As it is mentioned in the early sections of the paper, the solution of the LTWE contains both the upcoming and downgoing waves. However, when the velocity is constant in the medium (hence, the impedance is constant since the density is assumed to be constant in the full wave equation), the LTWE migration will not generate the reverberations on the final section. The non-reverberation solution of the LTWE, in the constant velocity medium, can be seen from the impulse response of the LTWE operator shown in Figure 4(a).

When velocity is smoothly varying, the reverberations generated by the LTWE operator is usually too small that they are negligible, comparing to the primary reflection images.

When the velocity function has a discontinuity at some depth, the waves extrapolated at that position will be transmitted and also reflected, generating the reverberations. The reverberation can be reduced if we ignore the velocity error term represented by equation (36). Figures 6(a) and 6(b) show the comparison between ignoring and adding the error term in the LTWE migration. The model has a sharp velocity discontinuity at τ=18: $v(\tau<19)=2$ and $v(\tau\geq 18)=4$ (units are normalized). The input data is a spike at ($t=28, x=33$). The reverberation appears as the downward curving curve on the middle of the figures marked by the symbol M.

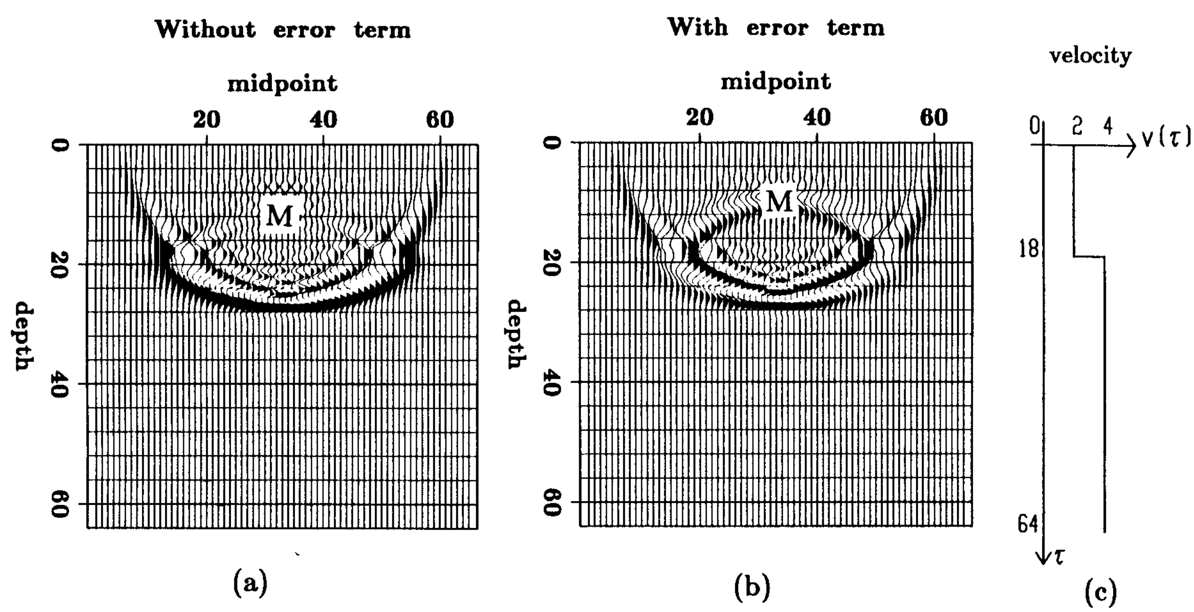

FIG. 6. The significance of the velocity error term in generating the reverberation from velocity discontinuity in the LTWE migration. (a) The impulse response without the velocity error term in the LTWE. (b) The impulse response with the velocity error term in the LTWE. (c) The velocity function of the model.

The finite-difference LTWE algorithm coded with the velocity error term can be used in modeling both the primary and the multiple reflections.

APPLYING THE LTWE MIGRATION TO THE FIELD DATA

The purpose of migration is to migrate both diffractions and dipping reflections. The following factors are used in deciding which algorithms to use for migrating a stacked section: economy of computation, purpose of migration, quality of stacked section, and geology of the section. No migration is needed if the section has only flat events. Lower-order equation algorithms can be used if no steeply dipping reflections are present in the stacked section. The quality of preserving large angle events in stacked section can be improved if dip-moveout, or constant velocity stack algorithms, or other algorithms, are used. The LTWE migration can give better results than do lower-order-equation migrations if the seismic data are recorded over an area where the geology is complicated and if the large-angle reflections are well stacked into the stacked sections.

The LTWE migration is used on a Chevron dataset of the Gulf of Mexico. The input data is obtained by *double slant stacking* over near traces of profiles. This idea of slant stacking over near traces was suggested by J. Claerbout and will be discussed in details in the next report. Stacking in the shot-geophone space better preserves the higher-angle reflections in the stacked section than does the conventional stacking in the midpoint-offset space, because the data aliasing problem is less severe in the shot-geophone space than in the midpoint-offset space. One part of double slant stacked section is shown in Figure 7. The reflections from steeply dipping fault planes are well stacked into the section for migration. Both the conventional 15 degree and the LTWE migration algorithms are used to migrate the stacked section.

In both the 15 degree and the LTWE migrations, a few traces at each side of the input data are tapered slightly so that the reflection from the edges of the finite-differencing grids is reduced.

The diffraction tails in the unmigrated section are shrunk, or collapsed into the diffracting points in the migrated sections shown on Figures 8(a) and 8(b). The fault plane reflection between horizontal coordinates 7.9 km and 10 km about 1.8 seconds to 3 seconds is better migrated and preserved by the LTWE method than the 15 degree one. The 15 degree equation migrated section almost losts the upper part of this fault plane reflection, because the upper part of the fault plane has larger angle than the lower part and the 15 degree migration method does not handle the large angle reflection properly. It is actually the dip-filter effect of the 15 degree operator that eliminates the higher-angle reflection. The 15 degree equation migration also tends to under-migrate the dipping events. Some under-migrated fault plane reflections on both the migrated sections are possibly caused by the out-of-plane reflection problem at those positions. Three-dimensional migration should migrate them properly.

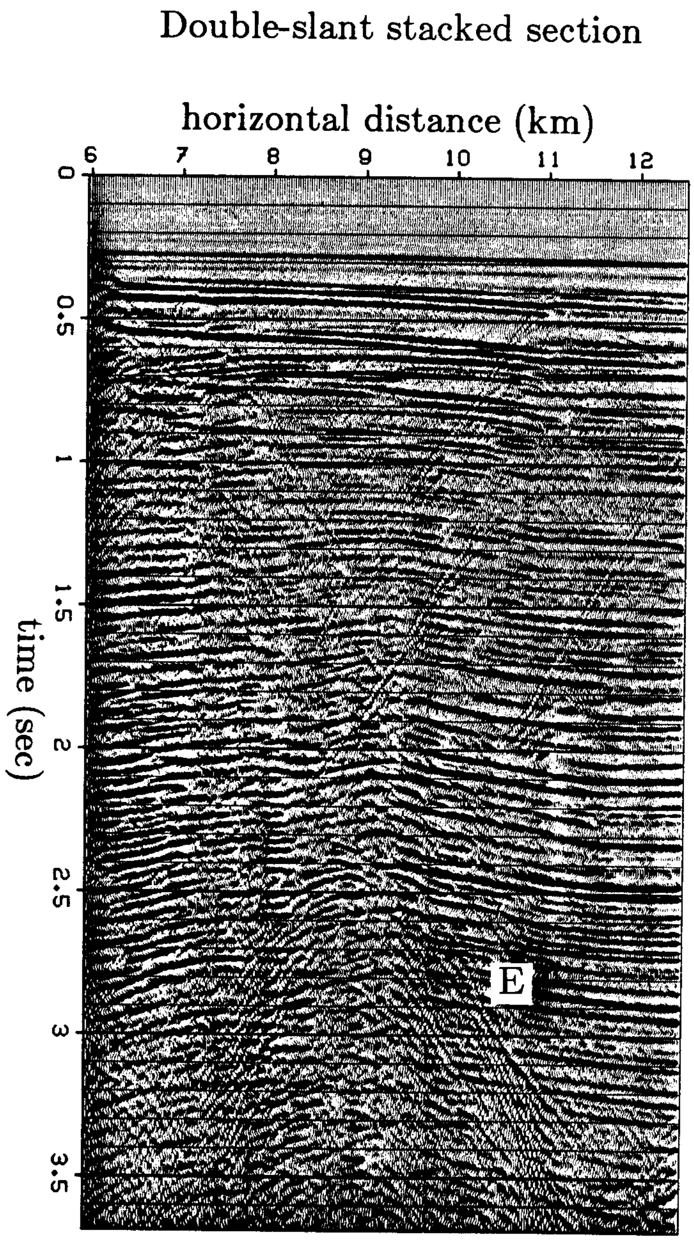

FIG. 7. The unmigrated double slant stacked section of Chevron dataset in the area of the Gulf of Mexico. The horizontal distance spacing is 25 meters. The time spacing in the sections is 0.008 second.

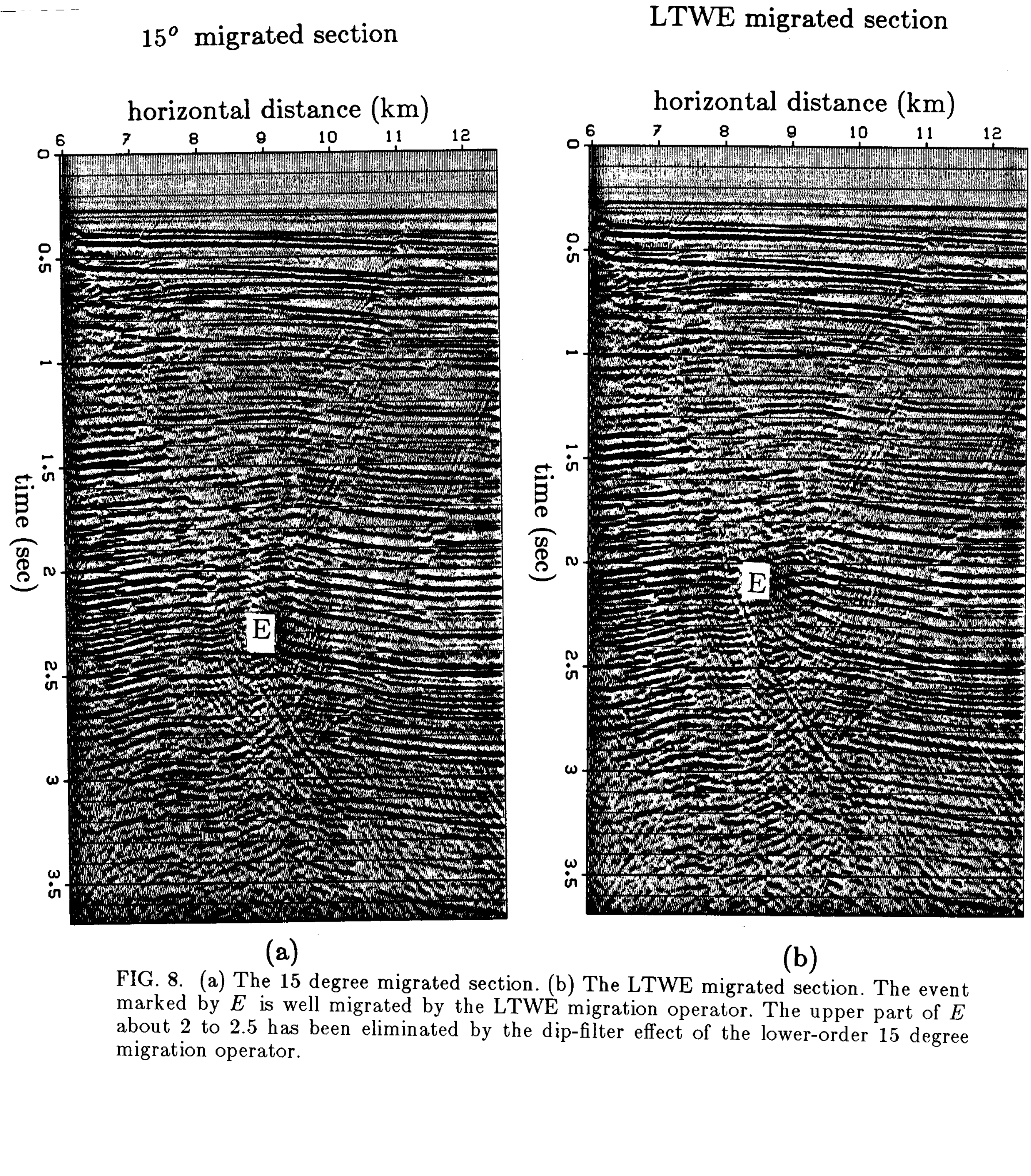

FIG. 8. (a) The 15 degree migrated section. (b) The LTWE migrated section. The event marked by E is well migrated by the LTWE migration operator. The upper part of E about 2 to 2.5 has been eliminated by the dip-filter effect of the lower-order 15 degree migration operator.

CONCLUSION

The LTWE equation is accurate for events with all possible angles of propagation. The linear transformation over both depth and time reduces the two terms of second derivative in the full wave equation to a single second cross derivative term in the LTWE, instead of dropping a second derivative as in the case of the retarded coordinate transform: $t' = t - z/v$, $z' = z$; or the moving coordinate transform: $z' = z + vt$, $t' = t$. It is actually a combination of these two transforms.

The Linear Transformation give us an accurate wavefield extrapolation operator which can be applied not only to stacked data migration and both primary and multiple reflection modeling, but also to prestack migration and three dimensional wavefield extrapolation, etc.

ACKNOWLEDGMENTS

I thank Professor J.F. Claerbout, who is guiding and inspiring my study and research at SEP, for bringing out the problem of imaging both sides of steeply dipping reflector which led me to find the LTWE in order to handle the dipping reflections. The persistent help and suggestions from R. Ottolini in my research at Stanford, is also very appreciated. I also thank J. Thorson, S. Levin, Professor F. Muir, W. Harlan and the other SEP senior students for commenting on the work and giving suggestions for the future work in this subject.

In the section of estimating the velocity error term in this paper, J. Thorson gave very first qualitative estimate which led me to find the mathematical representation. I appreciate J. Thorson's helpful discussion with me in this aspect. In the section of the upcoming wave boundary condition and the two way wave solution of the LTWE in this paper, I appreciate and enjoy the interesting discussions with S. Levin and W. Harlan about eliminating one solution from the two- wave equation solution. J. Claerbout, J. Thorson, R. Ottolini and J. Dellinger also contributed to the discussion of two solution problem in both the LTWE and the 15 degree equation. A Chinese saying best summarizes my acknowledgements, " *Putting heads together so as to get better results* ", which is equivalent to an English one " *Two heads are better than one* ".

P. Fowler and R. Ottolini proof read a draft of this paper and gave helpful suggestions in organizing the paper. Many thanks are also due to all the SEP sponsors for their support of SEP.

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APPENDIX

A RATional FORtran (RATFOR) subroutine of the LTWE migration

The subroutine does the LTWE migration. The wave equation used is given by equation (34) in (x, τ, t) space. The velocity independent linear transformation,

$$\begin{cases} t_1 = \frac{1}{\sqrt{2}} \left\{ \tau + t \right\} \\ t_2 = \frac{1}{\sqrt{2}} \left\{ \tau - t \right\} \\ x' = x \end{cases} , \tag{A-1}$$

is applied to equation (34) to obtain the LTWE. The velocity error term due to the vertical velocity gradient is also coded into the program.

The subroutine calls a real tridiagonal-matrix-solver subroutine (rtri) in each extrapolation step, because the velocity is assumed to vary point by point in (x, τ).

When the velocity does not change point by point, the calculation of coefficients of the matrix is necessary only when the differencing stars are moved to the region where velocity is different from the velocity in the previous region. When the velocity keeps the same for many steps of wavefield extrapolation, the Cholesky's method of solving tridiagonal systems of equations should be used instead of the method implemented in the subroutine, rtri. The Cholesky's method decomposes a tridiagonal matrix into a product of a lower bidiagonal matrix and an upper bidiagonal matrix. The coefficients of these two matrix can be saved and used again by the later steps (Atkinson, 1978).

```
#           The LTWE migration subroutine
#
# p         input data and output data
# nt,nx           trace length, number of traces
# dt,dx           sampling intervals
# v         velocity profiles of length nt; v=v(nt,nx)
# trick           1/6. trick in more accurate finite difference
#            approximation of second x derivative;
# ver=            1: velocity error term is taken into consideration
#

# Define nxmax and ntmax in the main program before using the subroutine
# where ntmax=nt  and nxmax=nx.
subroutine LTWE(p,v,trick,ver,nt,nx,dt,dx)
real p(nt,nx),u(nxmax),w(nxmax),z(nxmax),y(nxmax),d(nxmax),apb(nxmax)
real a,tmp,trick
real diag(nxmax,ntmax),v(nt,nx),dt,dx,dd,offdi(nxmax,ntmax),verror(nxmax),ver
integer ix,nx,it,nt,it1,it2,mt,it1end,sqrt2

tmp=dt*dt/(2*8*dx*dx*4)
sqrt2=sqrt(2.)
do iter = 1, 2
      {
      do it2=1,nt-1,1
            {
            do ix=1,nx
            {
            u(ix)=0.        # upcoming wave boundary condition
            z(ix)=0.
            }
     if (iter == 1)
            it1end=nt-it2+1 # for negative t2 :  -nt < t2 <= 0
      if (iter == 2)
            it1end=it2+1    # for positive t2 :  0 < t2 <= nt
```

```
do it1=nt,it1end,-1
        {
        # update the differencing star
        # Differencing star:  u=p(t1,t2+1)        w=p(t1+1,t2+1)
        #                     z=p(t1,t2)    y=p(t1+1,t2)
        do ix=1,nx
                {
                        w(ix)=u(ix)
                        y(ix)=z(ix)
                        z(ix)=p(it1,ix)
                }
        if (iter == 1)
                mt=(it1+it2+1-nt)/2        # locate the tau coordinate
        if (iter == 2)
                mt=(it1+it2)/2        # locate the tau coordinate

        do ix=1,nx    # calculate the tridiagonal matrix coefficients
        {
        a=v(mt,ix)*v(mt,ix)*tmp
        # a = v*v*dt1*dt2/(8*dx*dx)=tmp*v*v
        #dt1=dt/sqrt(2.) Transformed time interval dt1=dt2=dt/sqrt(2.)
        apb(ix) = a+trick;
        diag(ix) =1.+2.*(a-trick) # tridiagonal coefficients
       offdi(ix) = trick-a
        verror(ix) = 0.
        }
#  Error term in the wave equation under coordinate (tau,x,z), due to
#  velocity variation
        if (ver ==1)
                {
                do ix=1,nx
verror(ix)=(v(mt+1,ix)-v(mt,ix))/(sqrt2*v(mt,ix)*v(mt,ix)*v(mt,ix)*dt)
                }
        # compute right-hand-side column vector; zero-slope b.c.'s
        dd = (1.-verror(1)-apb(1))*(z(1)+w(1))+apb(2)*(z(2)+w(2))
        d(1) = dd-diag(1)*y(1)-offdi(1)*y(1)-offdi(2)*y(2)
        do ix=2,nx-1
                {
                dd    = (1.-verror(ix)-2.*apb(ix))*(z(ix)+w(ix))
                dd    = dd + apb(ix-1)*(z(ix-1)+w(ix-1))
                dd    = dd + apb(ix+1)*(z(ix+1)+w(ix+1))
```

```
                d(ix) = dd-diag(ix)*y(ix)-offdi(ix-1)*y(ix-1)
                d(ix) = d(ix) - offdi(ix+1)*y(ix+1)
                }
      dd =(1.-verror(nx)-apb(nx))*(z(nx)+w(nx))+apb(nx-1)*(z(nx-1)+w(nx-1))
            d(nx) = dd-diag(nx)*y(nx)-offdi(nx)*y(nx)-offdi(nx-1)*y(nx-1)
            # solve tridiagonal system; zero-slope boundary conditions
            diag(1)=diag(1)+offdi(1)
            diag(nx)=diag(nx)+offdi(nx)
call rtri(nx,offdi,diag,offdi,d,u)
            do ix=1,nx
                p(it1,ix) = u(ix)
            }
      }

}
return
end
#     subroutine, rtri,  solves a real tridiagonal system of equations :
#                 A X = D
#
#     b are the main diagonal coefficients
#     a are the lower diagonal coefficients
#     c are the upper diagonal coefficients
      subroutine rtri(nx,offdi,diag,offdi,d,u)
      real a(n),b(n),c(n),x(n),d(n),e(n),f(n)
      integer n,i
      real den
      e(1) = - c(1) / b(1)
      f(1) = d(1) / b(1)
      do i = 2,n-1
            {
            den =  1./(b(i) + a(i) * e(i-1))
            e(i) = -c(i) * den
            f(i) = ( (d(i) - a(i) * f(i-1)) ) * den
            }
      x(n) = (d(n)-a(n)*f(n-1))/(b(n)+a(n)*e(n-1))
      do i = n-1,1,-1
            x(i) = f(i) + e(i) * x(i+1)
      return
      end
```

вместо него используется неявная зависимость $F(\omega, k_x, k_y)$. В таком случае в соответствии с известным соотношением из теории частных производных имеем

$$\frac{\partial \omega}{\partial k_x} = -\frac{\partial F/\partial k_x}{\partial F/\partial \omega}.$$

В экспериментальной геофизике понятие скорости почти всегда связывается с групповой скоростью, с которой перемещается

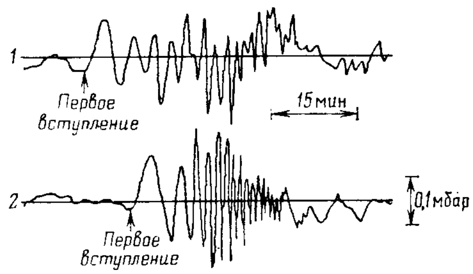

РИС. 1.11. Пример двух волн *1* и *2* звукового давления, которые считаются результатом ядерного взрыва в Азии (они были зарегистрированы в Калифорнии одним из микробарографов автора)

энергия. На рис. 1.11 присутствует излишний «шум» (обычный в экспериментальной геофизике), однако можно видеть, что возмущение проявляется сначала в виде колебаний с большим периодом, а затем уже в виде колебаний с коротким периодом. Групповая скорость находится путем деления расстояния на время распространения волны. Фазовые скорости можно изучать, имея две станции наблюдения недалеко друг от друга и измеряя временную задержку некоторых специфических пересечений нулевой линии. Причина, по которой станции располагают недалеко друг от друга, заключается в том, что форма волны непрерывно изменяется и если станции отстоят слишком далеко друг от друга, то не будет возможности определения тех пересечений нулевой линии, которые должны сравниваться.

1.5. КОРРЕЛЯЦИЯ И СПЕКТРЫ

Спектр временной функции есть возведенное в квадрат преобразование Фурье этой функции. В случае действительной функции преобразование Фурье имеет четную действительную часть RE и нечетную мнимую часть IO. Взяв квадраты модулей, имеем:

$$(\mathrm{RE}+i\,\mathrm{IO})(\mathrm{RE}-i\,\mathrm{IO}) = (\mathrm{RE})^2+(\mathrm{IO})^2.$$

Квадрат нечетной функции так же как и квадрат четной функции является, очевидно, четной функцией. Таким образом, спектр действительной функции времени есть четная функция, так что его значения в области положительных частот совпадают с его значениями в области отрицательных частот. Поэтому нет особого смысла выделять отрицательные частоты. Хотя большинство встречающихся на практике функций времени являются действительными, рассмотрение корреляции и спектров не будет математически полным без функций времени с комплексными значе-

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From the Russian translation of Jon Claerbout's *Fundamentals of Geophysical Data Processing*. The caption to Figure 1.11 reads: "An example of two waves, 1 and 2, of atmospheric pressure, which are believed to be the result of a nuclear explosion in Asia (they were recorded in California on one of the author's microbarographs)." No times or dates are given.