To prove that $\hat{C}$ is causal, we will take advantage of rule 1, namely, that an impedance can be scaled by any real positive number that you like, and it will still be an impedance. Consider a function that is similar to $\hat{C}$.

$$B \quad = \quad \frac{1 - \epsilon \hat{R}}{1 + \epsilon \hat{R}} \tag{22}$$

Choose ϵ small enough that for all ω, $\epsilon |\hat{R}| < 1$. This ensures a convergent expansion for the denominator in positive powers of $\hat{R}$ and hence Z. The expansion contains only positive powers in the delay operator. Thus B is a reflectance, and its corresponding impedance is $\epsilon \hat{R}$. But an impedance can always be scaled by a positive number. Taking the number to be $1/\epsilon$ shows that $\hat{R}$ is an impedance. This completes the proof that every CPR is an impedance.

So impedances arise more easily than you might think. It is not necessary to have a reflectance C to insert into the relation $R = (1-C)/(1+C)$. We only need to have a CPR.

Functional Analysis

We will establish the following theorems about exponentials, logarithms, and powers of Fourier transforms of filters:

1. The exponential of a causal filter is causal.
2. The exponential of a causal filter is a minimum-phase filter.
3. The logarithm of a minimum-phase filter is causal.
4. The Fourier domain representation of a minimum-phase filter is a curve that does not enclose the origin of the complex plane.
5. Any power of a minimum-phase filter is minimum phase.
6. Any real fractional power $-1 \le \rho \le 1$ of an impedance function is an impedance function.

To establish theorem 1, define the Z-transform of an arbitrary causal function

$$U(Z) \quad = \quad u_0 + u_1 Z + u_2 Z^2 + \cdots \tag{23}$$

and substitute it into the familiar power series for the exponential function:

$$B(Z) \quad = \quad e^U \quad = \quad 1 + U + \frac{U^2}{2!} + \frac{U^3}{3!} + \cdots \qquad (|U| < \infty) \tag{24}$$

No negative powers of Z can be found in the right side of (24), so $B(Z)$ will have no negative powers of Z. Also, the factorials in the denominator