Chapter III

Slant Stack Generalized Inverses

3.1 Introduction

To illustrate the points made in chapter 2, we shall derive the slant stack generalized inverse (or pseudoinverse) in this chapter. We shall see that the theorem given at the end of section 2.7 is valid when the operator **L** is chosen to be the slant stack. This theorem also allows the pseudoinverse filter to be replaced by a filter simpler in form and much easier to apply in practice: the so-called *rho filter*. In the last section of this chapter, the two filters are applied to the same synthetic data. As expected, the resultant outputs are identical.

3.2 Slant stacking

Recall from section 2.5 that the slant stack adjoint pair, without including for now any truncation and aliasing effects, is

Slant **L**: $$d(h,t) = \int_{-\infty}^{\infty} dp \; u(p, t - ph) \tag{3.1}$$

Slant $\mathbf{L}^T$: $$u(p,\tau) = \int_{-\infty}^{\infty} dh \; d(p, \tau + ph) \tag{3.2}$$

The index h (offset) has dimensions of length, and p (slowness) has dimensions of inverse velocity. Typical sample rates in time on a gather are sufficiently high to preclude any problems with aliasing on the time axis. The sampling in offset, however, is commonly coarse and restricted to a narrow range of offsets.

A data set that consists of a number of coherent events of constant dip, can be accurately modeled by equation (3.1). Furthermore, events of interest on the data set will tend to lie within a limited range of dips, so that the function $u(p,\tau)$, which models the data via (3.1), may be assumed to be limited in p:

$$u(p,\tau) = 0 \qquad \text{for } p < p_1 \text{ and } p > p_2 \tag{3.3}$$

Recovering **u** from the incomplete data **d** is the *type I* problem of section 2.6: to complete the formulation of this problem, projections $\mathbf{P}_u$ and $\mathbf{P}_d$ must be defined. The constraint (3.3) implies the following definition for the projection $\mathbf{P}_u$:

$$P_u(p,\tau) = \begin{cases} 1 & p_1 \le p \le p_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

The constraint (3.3) is now equivalent to the statement $\mathbf{P}_u\mathbf{u} = \mathbf{u}$. However, under this assumption, the output $d(h,t)$ of equation (3.1) can never be bounded in h; that is, operators $\mathbf{L}$ and $\mathbf{L}^T$ are unbounded operators. Assuming that the sampling of the data in h is adequately dense but limited to the range $h_1 \le h \le h_2$, the projection operator $\mathbf{P}_d$ must be defined as

$$P_d(h,t) = \begin{cases} 1 & h_1 \le h \le h_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

With these definitions the slant stack operators $\mathbf{L}$ and $\mathbf{L}^T$ can be converted into bounded operators by premultiplication and postmultiplication with $\mathbf{P}_d$ and $\mathbf{P}_u$. If assumption (3.3) is made, it is not necessary to explicitly carry around the term $\mathbf{P}_u$; from now on it will be dropped. The bounded operators $\mathbf{P}_d\mathbf{L}$ and $\mathbf{L}^T\mathbf{P}_d$ differ from those defined in equations (3.1) and (3.2) only in the limits imposed on the integrals:

$\mathbf{P}_d\mathbf{L}$: $$d(h,t) = \int_{p_1}^{p_2} dp \; u(p, t - ph) \tag{3.6}$$

$\mathbf{L}^T\mathbf{P}_d$: $$u(p,\tau) = \int_{h_1}^{h_2} dh \; d(p, \tau + ph) \tag{3.7}$$

Recall that the *type I* pseudoinverse defined in section 2.6 is $\mathbf{u} = (\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+\mathbf{L}^T\mathbf{P}_d\mathbf{d}$. With the slant stack operators of (3.6) and (3.7), it can be implemented in two steps: first stack the data with $\mathbf{L}^T\mathbf{P}_d$, then apply $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$. This last term could just as well be called the pseudoinverse; in fact it is exactly the pseudoinverse of the operator $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$.

3.3 Calculating the slant stack pseudoinverse

It remains to determine $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ and its pseudoinverse $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$. The pseudoinverse can be found because of the simple structure imposed on the truncator $\mathbf{P}_d$: the orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ of the singular value decomposition of $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ turn out to be Fourier transforms.

The response $\mathbf{Lu}$ to an impulse $u(p,\tau) = \delta(p - \tilde{p})\delta(\tau - \tilde{\tau})$ is

$$\begin{aligned} d(h,t) &= \int_{-\infty}^{\infty} dp \; \delta(p - \tilde{p})\delta(t - ph - \tilde{\tau}) \\ &= \delta(t - \tilde{p}h - \tilde{\tau}) \end{aligned} \tag{3.8}$$

Applying $\mathbf{L}^T\mathbf{P}_d$ to this gives the impulse response of $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$:

$$\begin{aligned} u(p,\tau) &= \int_{h_1}^{h_2} dh \; \delta(\tau + ph - \tilde{p}h - \tilde{\tau}) \\ &= \int_{h_1}^{h_2} dh \; \frac{\delta(h - h_0)}{|p - \tilde{p}|} \end{aligned} \tag{3.9}$$

where $h_0 \equiv -(\tau-\tilde{\tau})/(p-\tilde{p})$. Thus

$$u(p,\tau) = \begin{cases} |p-\tilde{p}|^{-1} & \text{for } h_1 \le h_0 \le h_2 \\ 0 & \text{otherwise} \end{cases}$$

$$= |p-\tilde{p}|^{-1} H(h_0-h_1)\, H(h_2-h_0) \qquad (3.10)$$

where $H(x)$ is the Heaviside, or unit step function. With this result the transformation $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ may be represented in the form of a double integral with kernel K:

$$u(p,\tau) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\tau\, K(p,\tau;\tilde{p},\tilde{\tau})\, \tilde{u}(\tilde{p},\tilde{\tau}) \qquad (3.11)$$

in which the kernel is

$$K(p,\tau;\tilde{p},\tilde{\tau}) = |p-\tilde{p}|^{-1}\, H\left(-\frac{\tau-\tilde{\tau}}{p-\tilde{p}}-h_1\right) H\left(h_2+\frac{\tau-\tilde{\tau}}{p-\tilde{p}}\right) \qquad (3.12)$$

Note that the kernel K of the filter is convolutional in both τ and p, and therefore the filter is multiplicative in two-dimensional Fourier transform (2DFT) space. In other words, the operator $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ is diagonalized by a 2DFT. In the present context it is important that the forward and inverse Fourier transform pair be adjoints of each other: the singular value decomposition of $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ into $\mathbf{U}\Sigma\mathbf{V}^T$ requires that $\mathbf{U}$ and $\mathbf{V}^T$ be an adjoint pair. The Fourier transform convention used here is

$$\tilde{u}(\xi,\eta) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\tau\, u(p,\tau)\, e^{-i\xi p-i\eta\tau}, \qquad (3.13a)$$

$$u(p,\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, \tilde{u}(\xi,\eta)\, e^{+i\xi p+i\eta\tau}. \qquad (3.13b)$$

We shall identify the forward 2DFT with the operator $\mathbf{U}^T$, and the inverse 2DFT with the operator $\mathbf{U}$.

To find the filter of equation (3.12) in the Fourier domain, Fourier transform the kernel $K(p,\tau)$:

$$\tilde{K}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\ d\tau\ |p|^{-1} H\left(-\frac{\tau}{p} - h_1\right) H\left(h_2 + \frac{\tau}{p}\right) e^{-i\xi p - i\eta\tau}$$

$$= \frac{1}{2\pi}\left\{ \int_{0}^{\infty} dp \int_{-h_2 p}^{-h_1 p} d\tau + \int_{-\infty}^{0} dp \int_{-h_1 p}^{-h_2 p} d\tau \right\}\left\{ |p|^{-1} e^{-i\xi p - i\eta\tau} \right\} \quad (3.14)$$

Partitioning the integral in this way keeps the differential area under the double integral positive. Now let $\tau = hp$, $d\tau = p\ dh$. By this change of variables the integral becomes

$$\tilde{K}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \int_{-h_2}^{-h_1} dh\ |p|\ |p|^{-1} e^{-i\xi p - i\eta p h}$$

$$= \int_{-h_2}^{-h_1} dh\ \delta(\xi + \eta h) = \int_{-h_2}^{-h_1} dh\ \frac{\delta(\xi/\eta + h)}{|\eta|} \quad (3.15)$$

Therefore

$$\tilde{K}(\xi,\eta) = |\eta|^{-1} H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right) \quad (3.16)$$

where $H(x)$ again is the unit step function, and is used to define the region where the filter is nonzero: that is, where the delta function of equation (3.15) lies within the finite bounds of the integral. The nonzero region of the filter $\tilde{K}(\xi,\eta)$ in the Fourier plane is shown in figure 3.1. As $h_1 \to -\infty$ and $h_2 \to \infty$, the filter covers the entire Fourier plane.

An extra factor of 2π arises as a consequence of our choice for the symmetric Fourier transform pair (3.13). The transform of $\delta(x)$ is now $1/(2\pi)$, and the convolution rule, which normally has no scaling factors present, is modified to become

$$f(x) * g(x) \overset{FT}{\longleftrightarrow} \sqrt{2\pi}\, F(\xi)\, G(\eta) \quad (3.17)$$

Another explanation for the extra factor lies in the fact that, in order to properly transform an operator into another domain, a similarity transformation $\tilde{\mathbf{K}} = \mathbf{U}^T\mathbf{K}\mathbf{U}$ is required. A Fourier transform must therefore be performed over both the input and

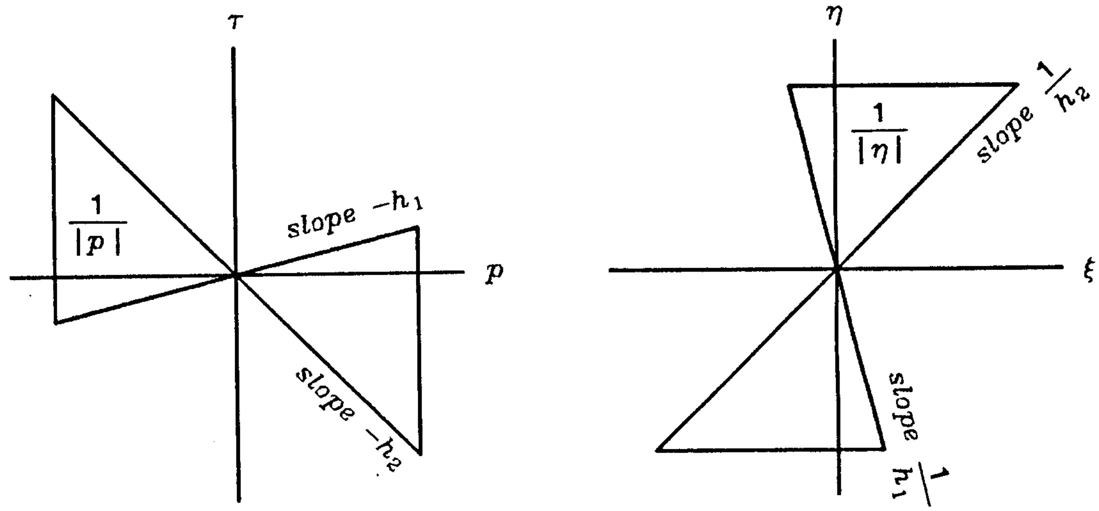

FIG. 3.1. Slant stack impulse response $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ in the (h,t) domain (equation 3.12) and the filter response in the Fourier domain (equation 3.16). The filter in Fourier space is nonzero within the range of dips that are reciprocal to the dip limits in the time-space domain.

output variables of the operator's kernel. When the kernel happens to be convolutional (as **K** is in our case) this transformation simplifies to:

$$\tilde{K}(\tilde{\xi}, \xi) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} dx\, d\tilde{x}\, f(\tilde{x} - x)\, e^{i\xi x - i\tilde{\xi}\tilde{x}}$$

$$= \int_{-\infty}^{\infty} dx'\, f(x')\, e^{-i\tilde{\xi}x'}\, \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-i(\tilde{\xi} - \xi)x}$$

$$= \sqrt{2\pi}\, F(\tilde{\xi})\, \delta(\tilde{\xi} - \xi) \qquad (3.18)$$

where $F(\xi)$ is the one-dimensional Fourier transform of $f(x)$. The formal kernel of $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ in the Fourier domain is found by applying rule (3.18) to equation (3.16):

$$\tilde{K}(\tilde{\xi},\xi;\, \tilde{\eta},\eta) = \frac{2\pi}{|\eta|} H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right) \delta(\tilde{\xi} - \xi)\, \delta(\tilde{\eta} - \eta) \qquad (3.19)$$

In short, when such a kernel is transformed, an extra factor of 2π must be included with the forward 2DFT; likewise $1/2\pi$ must be included with the inverse 2DFT.

The pseudoinverse is implemented by taking the inverse of the nonzero portion of the filter in the Fourier domain. In this case the null space of the operator $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ is obvious: it comprises events whose dip spectrum lies outside the nonzero range of the filter in figure 1. The pseudoinverse of $\widetilde{K}$ is thus

$$\widetilde{K}^+(\xi,\eta) \;=\; \frac{|\eta|}{2\pi} H\left[\frac{\xi}{\eta} - h_1\right] H\left[h_2 - \frac{\xi}{\eta}\right] \tag{3.20}$$

As the aperture (h_1, h_2) widens, the filter converges to $|\eta|/2\pi$, which is the so-called *rho filter*.

The term "rho filter" is borrowed from algebraic reconstruction theory (Swindell and Barrett, 1977). The only difference between algebraic reconstruction and our case, slant stacking, lies in the choice of coordinate system; the equations for algebraic reconstruction are derived in a polar coordinate frame instead of a slant frame. It is not surprising that the rho filter, which is applied before the back projection (or stacking) step, has the same analytic form $|\rho|$ as equation (3.20), in which ρ is a spatial wavenumber (Macovski, 1983, p. 127).

A time-space domain implementation of this filter is preferable, for in practice the spatial axis is discrete and limited. An implementation of the filter in the spatial wavenumber domain will result in serious wraparound problems. It is better to find an expression for $\widetilde{K}^+$ in the (p,τ) domain:

$$K^+(p,\tau) \;=\; \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\eta\, d\xi\, \frac{|\eta|}{2\pi} H\left[\frac{\xi}{\eta} - h_1\right] H\left[h_2 - \frac{\xi}{\eta}\right] e^{ip\xi+i\tau\eta} \tag{3.21}$$

Again let $h = \xi/\eta$ be the new variable of integration so that $\xi = \eta h$ and $d\xi = |\eta|\, dh$. The limits of the integral are divided in such a manner to guarantee a positive area differential:

$$K^+(p,\tau) = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} d\eta \int_{h_1}^{h_2} dh\ \eta^2\ e^{i\eta(\tau + hp)}$$

$$= \frac{1}{8\pi^3}\int_{-\infty}^{\infty} d\eta\ \frac{-i\eta}{p} e^{i\eta\tau}\left[e^{i\eta p h_2} - e^{i\eta p h_1}\right]$$

$$= \frac{1}{8\pi^3}\int_{-\infty}^{\infty} \frac{-i\eta}{p} e^{i\eta(\tau + ph_2)} d\eta \ - \ \frac{1}{8\pi^3}\int_{-\infty}^{\infty} \frac{-i\eta}{p} e^{i\eta(\tau + ph_1)} d\eta \qquad (3.22)$$

or,

$$K^+(p,\tau) = \frac{1}{4\pi^2 p}\left[\delta'(\tau + ph_2) - \delta'(\tau + ph_1)\right] \qquad (3.23)$$

This filter, the kernel of the pseudoinverse, consists of a delta derivative positioned along the slopes $-h_2$ and $-h_1$, with a weight of $1/|p|$ applied. One way it could be implemented is by a finite differencing where the delta derivative is positioned. In a way, the inverse filter uses only the truncation effects associated with the forward filter K to do its work. Because the delta derivatives are positioned along the edges of the filter, the inverse kernel K^+ in equation (3.23) will detect variations of the input along the slopes $-h_1$ and $-h_2$. A frequency-bandlimited version of the impulse response of equation (3.23) is shown in figure 3.2.

When an infinite aperture is assumed, $h_1 \to -\infty$ and $h_2 \to \infty$, and the filter $K^+(p,\tau)$ reduces to the familiar one-dimensional rho filter. This reduction is made by performing an inverse Fourier transform on the expression (3.20), with the unit step functions removed from the integrand:

$$K^+(p,\tau) = \frac{\delta(p)\,\rho(\tau)}{4\pi^2} \qquad (3.24)$$

where $\rho(\tau)$ is the inverse transform of $|\eta|$. In a strict sense, the inverse transform of $|\eta|$ is unbounded, but discrete approximations to it are bounded and are easily derived. Let us now give two possible discrete implementations of the rho filter.

One obvious way to design a discrete rho filter is to perform a discrete inverse Fourier transform on $|\eta|$:

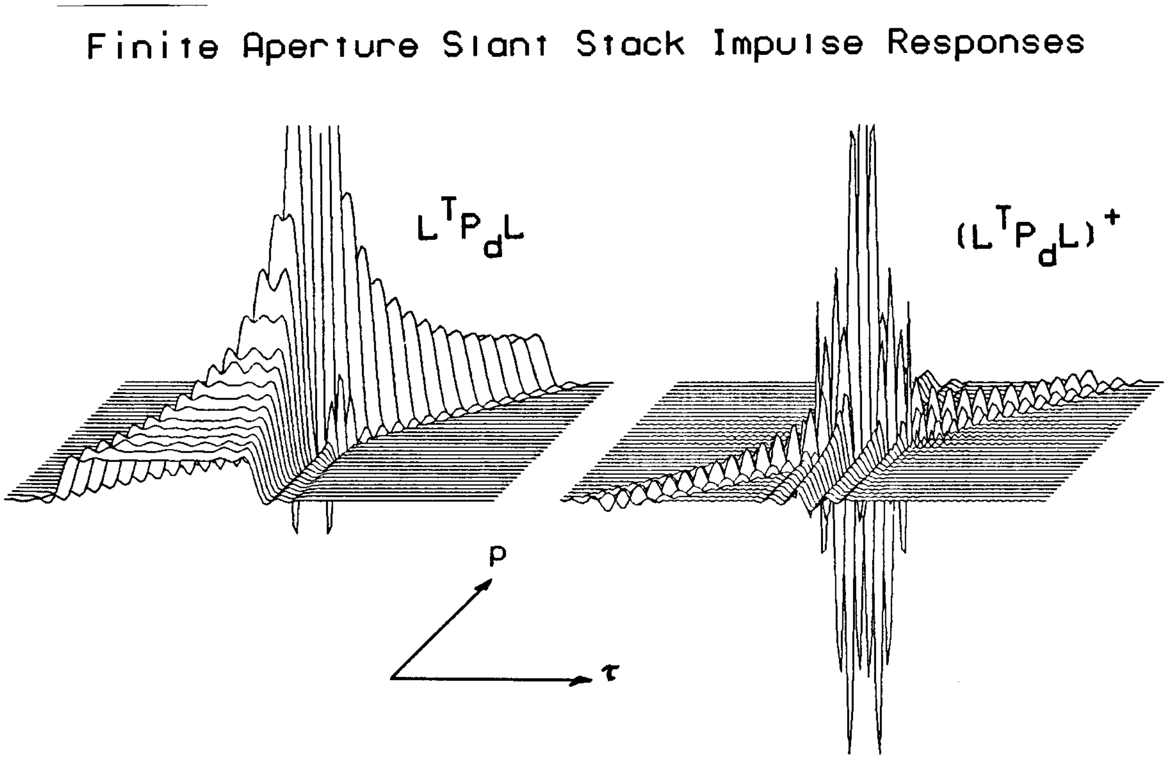

FIG. 3.2. Responses of $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ and $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$ to a sinc function (i.e., a bandlimited impulse in time). The ideal impulse responses in the (p,τ) domain are given by equations (3.12) and (3.23).

$$\rho(\tau) = \int_{-\pi/\Delta\tau}^{\pi/\Delta\tau} d\eta \; |\eta| e^{i\eta\tau} \tag{3.25}$$

The limits of integration are bounded by the temporal Nyquist frequency $\pi/\Delta\tau$. Performing this transform yields

$$\rho(\tau) = \begin{cases} -\dfrac{4}{\Delta\tau^2 j_\tau{}^2} & j_\tau \; even \\ 0 & j_\tau \; odd \\ \dfrac{2\pi^2}{\Delta t^2} & j_\tau \; zero \quad (\tau = j_\tau\Delta\tau) \end{cases} \tag{3.26}$$

The alternative way to design the discrete rho filter is to calculate the continuous transform at those τ values where it exists. In this case, the integral is bounded everywhere except at $\tau = 0$.

$$\rho(\tau) = \int_{-\infty}^{\infty} d\eta \, |\eta| e^{i\eta\tau} = \frac{-2}{\tau^2} \qquad \tau = j\Delta t \neq 0 \tag{3.27}$$

Because the filter $|\eta|$ can have no zero-frequency component, the remaining unknown filter element at $\tau = 0$ must be chosen to be the negative average of the other filter components:

$$\rho_0 = \sum_{j \neq 0} \Delta\tau \, \frac{2}{j^2 \Delta\tau^2} . \tag{3.28}$$

In summary, identifying the null space of $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ allows one to determine which events are unrecoverable from the slant stack inversion. Specifically, those events, whose slopes are not within the range of slopes that were stacked over, are eliminated in the forward stack and cannot be recovered by the pseudoinverse. Except for these events, the original data set may be reconstructed by the following process:

(a) Given $\mathbf{d}$, apply the transpose slant stack $\mathbf{L}^T\mathbf{P}_d\mathbf{d}$.

(b) Apply the pseudoinverse $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$.

Table 3.1 summarizes the forward and inverse filters developed in this section. For completeness, the operator $\mathbf{L}\mathbf{P}_u\mathbf{L}^T$ and its pseudoinverse are also included in the table. In this case the projector $\mathbf{P}_u$ limits the events' slopes p to lie between p_1 and p_2.

3.4 Equivalence of the finite offset pseudoinverse and the rho filter

Let us return to the theorem at the end of section 2.7. It states that under certain conditions the pseudoinverse $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+\mathbf{L}^T\mathbf{P}_d$ is equivalent to $(\mathbf{L}^T\mathbf{L})^+\mathbf{L}^T\mathbf{P}_d$. If these conditions hold, the projection $\mathbf{P}_d$ may be eliminated from the pseudoinverse. The projection $\mathbf{P}_u$ in section 2.7 can also be safely ignored, because of the assumption of equation (3.3). In any case the following results may be generalized

Slant Stack Filters			
Filter	Dip Range	Space Domain	Fourier Domain[1]
$\mathbf{L}^T\mathbf{P}_d\mathbf{L}$	h_1, h_2	$\lvert p\rvert^{-1} H\left(-\frac{\tau}{p} - h_1\right) H\left(h_2 + \frac{\tau}{p}\right)$	$\frac{2\pi}{\lvert\eta\rvert} H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right)$
	∞	$\lvert p\rvert^{-1}$	$\frac{2\pi}{\lvert\eta\rvert}$
$(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$	h_1, h_2	$\frac{1}{4\pi^2 p}\left(\delta'(\tau + ph_2) - \delta'(\tau + ph_1)\right)$	$\frac{\lvert\eta\rvert}{2\pi} H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right)$
	∞	$\frac{-1}{2\pi^2\tau^2}\delta(p)$ (note 2)	$\frac{\lvert\eta\rvert}{2\pi}$
$\mathbf{L}\mathbf{P}_u\mathbf{L}^T$	p_1, p_2	$\lvert h\rvert^{-1} H\left(-\frac{t}{h} - p_1\right) H\left(p_2 + \frac{t}{h}\right)$	$\frac{2\pi}{\lvert\omega\rvert} H\left(-\frac{k}{\omega} - p_1\right) H\left(p_2 + \frac{k}{\omega}\right)$
	∞	$\lvert h\rvert^{-1}$	$\frac{2\pi}{\lvert\omega\rvert}$
$(\mathbf{L}\mathbf{P}_u\mathbf{L}^T)^+$	p_1, p_2	$\frac{1}{4\pi^2 h}\left(\delta'(t - hp_2) - \delta'(t - hp_1)\right)$	$\frac{\lvert\omega\rvert}{2\pi} H\left(-\frac{k}{\omega} - p_1\right) H\left(p_2 + \frac{k}{\omega}\right)$
	∞	$\frac{-1}{2\pi^2 t^2}\delta(h)$ (note 2)	$\frac{\lvert\omega\rvert}{2\pi}$

Notes:
1. Fourier variables: $\tau \to \eta$, $p \to \xi$; $t \to \omega$, $h \to k$

2. The filter point at zero t or τ is taken to cancel the mean of the filter.

TABLE 3.1. Slant stack filters. $H(t)$ is the unit step function. For each filter, the impulse response in the time-space domain and its transform in the Fourier domain is given. An infinite dip range means $h_1, p_1 \to -\infty$ and $h_2, p_2 \to +\infty$. Superscript '+' refers to the pseudoinverse.

to include the effect of $\mathbf{P}_u$, by substituting $\mathbf{L}\mathbf{P}_u$ for $\mathbf{L}$ and $\mathbf{P}_u\mathbf{L}^T$ for $\mathbf{L}^T$. Recall that the theorem of section 2.7 holds when $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ and $\mathbf{L}^T\mathbf{L}$ share eigenspaces; in other words when the same unitary transformation diagonalizes $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ and $\mathbf{L}^T\mathbf{L}$. For the present case the transformation that does this is a familiar one: the 2D Fourier transform from (p,τ) to (ξ,η). The filter $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ in table 3.1 is a *finite aperture* filter, the term "aperture" referring to the limitation in offset $h_1 \le h \le h_2$. This aperture actually defines $\mathbf{P}_d$ in equation (3.5). As the aperture widens to infinity, $\mathbf{P}_d$ approaches the identity operator and $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ approaches $\mathbf{L}^T\mathbf{L}$. Therefore $\mathbf{L}^T\mathbf{L}$ and $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ may both be applied in the Fourier domain, and because they share common eigenvectors (the Fourier kernel), they satisfy the requirements of the theorem.

It is enlightening to examine more closely the reason why $\mathbf{L}^T\mathbf{L}$ and $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ have the same eigenvectors. The notation of section 2.7 shows that $\mathbf{L}$ may be represented by its singular value decomposition $\mathbf{V}\Sigma\mathbf{U}^T$. $\mathbf{U}^T$ has already been identified with the forward two-dimensional Fourier transform from (p,τ) space to, say, (ξ,η) space. But what is $\mathbf{V}\Sigma$? Since $\mathbf{V}$ is a member of the SVD, it must have orthonormal columns: $\mathbf{V}^T\mathbf{V} = \mathbf{I}$; also, Σ must simply be a multiplier with positive values. An equivalent formula for the singular value decomposition is given by $\mathbf{LU} = \mathbf{V}\Sigma$. Let us now determine $\mathbf{LU}$ by Fourier transforming operator $\mathbf{L}$ (given by equation 3.6) over the input space:

$$\begin{aligned} d(h,t) &= \int_{-\infty}^{\infty} dp\; u(p,\, t - ph) \\ &= \iint_{-\infty}^{\infty} dp\; d\tau\; \delta(t - ph - \tau)\, \frac{1}{2\pi} \iint_{-\infty}^{\infty} d\xi\, d\eta\; \tilde{u}(\xi,\eta)\, e^{i\xi p + i\tau\eta} \\ &= \iint_{-\infty}^{\infty} d\xi\, d\eta\; \tilde{u}(\xi,\eta)\, e^{i\eta t}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} dp\; e^{ip(\xi - \eta h)} \\ &= \int_{-\infty}^{\infty} d\eta\; \tilde{u}(\eta h,\, \eta)\, e^{i\eta t} \end{aligned} \tag{3.29}$$

The operator in equation (3.29) may be identified with $\mathbf{V}\Sigma$. The form of Σ is already known: it is simply the kernel $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ expressed in the Fourier domain (equation 3.19). In operator notation, $\mathbf{U}^T(\mathbf{L}^T\mathbf{P}_d\mathbf{L})\mathbf{U} = \Sigma^2$, and by comparing this to equation (3.19), we must conclude that Σ (ignoring the unit step functions $H()$) is the multiplier

$$\Sigma(\xi,\eta;\tilde{\xi},\tilde{\eta}) = \left|\frac{2\pi}{\eta}\right|^{1/2} \delta(\eta - \tilde{\eta})\,\delta(\xi - \tilde{\xi}) \tag{3.30}$$

Given this expression for Σ and the expression for $\mathbf{V\Sigma}$ in equation (3.29), the kernel for **V** must be

$$V(h,t;\,\xi,\eta) = \left|\frac{\eta}{2\pi}\right|^{1/2} \delta(\xi - \eta h)\, e^{i\eta t} \tag{3.31}$$

V consists of a stretch of the wavenumber axis $\xi = \eta h$ followed by a Fourier transform from η to t. It may be easily verified that **V** is normalized and thus satisfies the orthogonality requirement for the singular value decomposition: $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

The condition that $\mathbf{L}^T\mathbf{L}$ and $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ share eigenvectors **U**, was shown in section 2.7 to be equivalent to the requirement that $\mathbf{P}_d\mathbf{V} = \mathbf{VP}$ for some projection **P**. According to this requirement $\mathbf{P}_d$ (given by equation 3.5) must be able to pass through **V** (given by equation 3.31) and, though transformed, still retain the qualities of a projection operator. Because $\mathbf{P}_d$ is independent of time t, it commutes with the Fourier transform component of **V**:

kernel of $\mathbf{P}_d\mathbf{V}$:
$$\begin{aligned} & P_d(h)\left|\frac{\eta}{2\pi}\right|^{1/2} \delta(\xi - \eta h)\, e^{i\eta t} \\ &= \left|\frac{\eta}{2\pi}\right|^{1/2} \delta(\xi - \eta h)\, e^{i\eta t}\, P_d(\xi/\eta) \end{aligned} \tag{3.32}$$

P thus equals the projection $P_d(\xi/\eta)$, which is comprised of the two Heaviside unit step terms:

$$P(\xi,\eta) = H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right) \tag{3.33}$$

The fact that equation (3.33) describes a projection proves the validity of the relation

$$\mathbf{P}_d\mathbf{V} = \mathbf{VP} \tag{3.34}$$

in which **P** and $\mathbf{P}_d$ are both projections. As a consequence, the eigenvectors of $\mathbf{L}^T\mathbf{L}$ and $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ are equivalent, and the theorem of section 2.7 may be applied for the case of slant stacking. Relation (3.34) will also be used in chapter 4 to justify the application of the theorem of section 2.7 to velocity stacks.

3.5 Illustrations of pseudoinverse filtering

Because $|\eta|$ is the expression the rho filter $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$ assumes in the Fourier transform domain when $\mathbf{P}_d = \mathbf{I}$, it can be referred to as the *full aperture* rho filter. The finite aperture filter $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$ given in table 3.1 is more costly to implement, because its kernel is two-dimensional; the full-aperture rho filter can be implemented as a one-dimensional filter. Note that regardless of the size of the offset aperture (h_1, h_2), the pseudoinverse $\mathbf{u}$ of $\mathbf{d}$ is given as

$$\mathbf{u} = (\mathbf{L}^T\mathbf{L})^+\mathbf{L}^T\mathbf{P}_d\mathbf{d} \tag{3.35}$$

Equation (3.35) describes a valid means to compute the pseudoinverse $\mathbf{u}$ for any definition of the projection $\mathbf{P}_d$, as long as that projection satisfies the following conditions: $P_d(h,t)$ is independent of t; and $P_d(h)$ equals zero or one; i.e., $\mathbf{P}_d$ must of course remain a projection. Under these conditions $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ remains a time and space invariant filter, and is "diagonalized" by a Fourier transform. Therefore, the full aperture rho filter applied to $\mathbf{L}^T\mathbf{P}_d\mathbf{d}$ still yields the pseudoinverse, particularly when the data set $\mathbf{d}$ is discretely or otherwise arbitrarily sampled in offset. According to our theory $\mathbf{d}$ is still constrained to be continuously sampled in time, but in practice it is adequate to have $\mathbf{d}$ finely sampled.

Figure 3.3 demonstrates the equivalence of the rho filter with its finite aperture version. Panel (a) is the response $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$ to a bandlimited impulse $\delta(p - p_o)\,\mathrm{sinc}(\tau - \tau_o)$. It is thus the slant stack of a single dipping event with a $\mathrm{sinc}(\tau)$ waveform. $\mathbf{P}_d$ is defined by the finite offset range of the slant stack, $0 \le h \le h_{\max}$. Panels (b) and (c) are the results of applying $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$ and $(\mathbf{L}^T\mathbf{L})^+$, respectively, to the slant stacked bandlimited impulse shown in panel (a). There is no perceivable difference between panels (b) and (c). Panels (e) and (f) are a

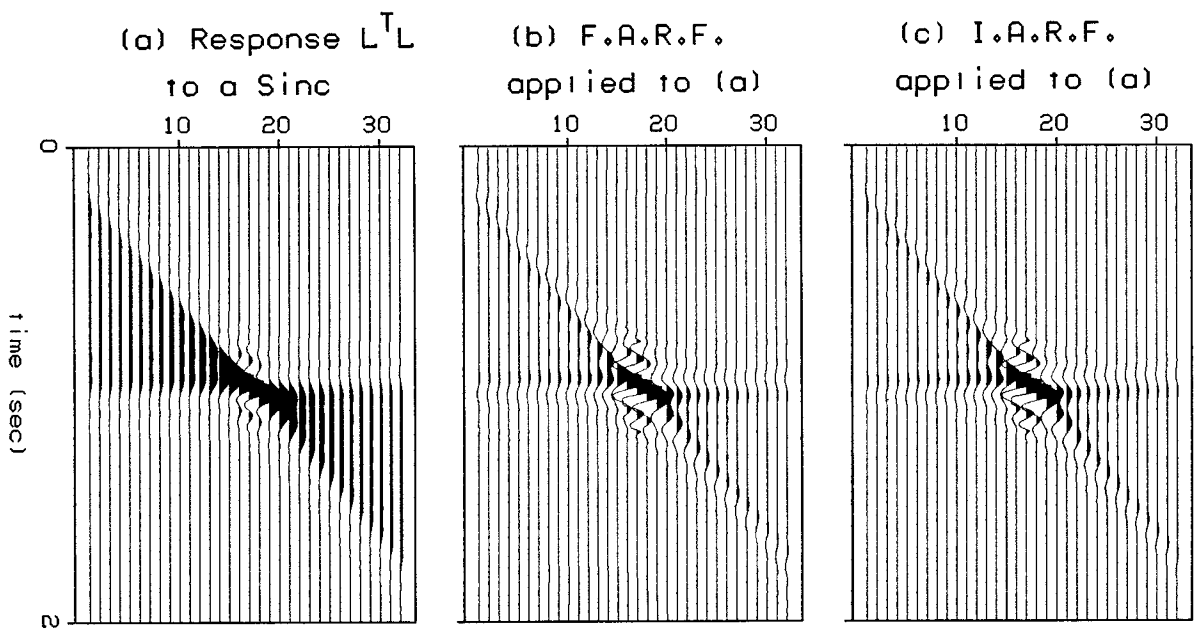

FIG. 3.3. The resolving kernels of the pseudoinverse.

(a) The data set **d**, which is not shown, consists of a zero phase, bandlimited wavelet with linear moveout. Consequently the data set is a slant stack of a single event centered in the (p,τ) plane shown. Only positive offsets are present in the data domain. Applying the transpose slant stack to **d** gives the response in panel (a), which is another illustration of the impulse response of $\mathbf{L}^T\mathbf{P}_d\mathbf{L}$.

(b) Finite aperture rho filter $(\mathbf{L}^T\mathbf{P}_d\mathbf{L})^+$ applied to panel (a). See table 3.1 for its definition.

(c) Infinite aperture rho filter $(\mathbf{L}^T\mathbf{L})^+$ applied to panel (a). This is the "standard" rho filter $|\eta|$ given in table 3.1.

similar comparison in which the impulse response in panel (d) has been corrupted with independent additive Gaussian noise. Again, there is no difference in applying the full aperture filter or the finite aperture filter. The results of the two operations are the same.

A measure of how well the pseudoinverse performs as an ideal slant stack inversion is given by the degree of focusing of the impulse response in figure 3.3. In panels (b) and (c) the spread of the impulse response (the sidelobes) has been significantly reduced with respect to the response in panel (a).

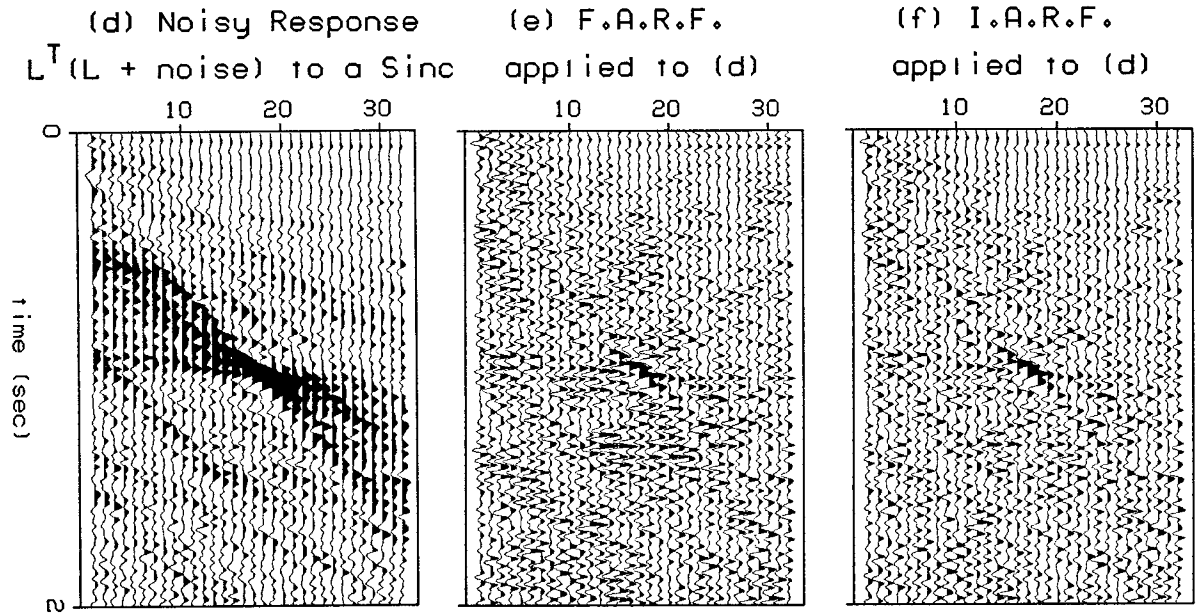

FIG. 3.3. (continued)

(d) Same as panel (a), except some zero mean independent noise was added to **d** before slant stacking.

(e) Finite aperture rho filter $(\mathbf{L}^T \mathbf{P}_d \mathbf{L})^+$ applied to panel (d).

(f) Infinite aperture rho filter $(\mathbf{L}^T \mathbf{L})^+$ applied to panel (d).

3.6 Summary

The expressions for the slant stack pseudoinverses, which are summarized in table 3.1, have been relatively easy to derive because of the known form that the singular value decomposition takes: a simple fan-filtering operation in the Fourier domain. The examples in the last section also illustrate the equivalence, in practice, of the finite-aperture and infinite-aperture rho filters, the equivalence being theoretically proved with the use of the theorem of section 2.7. The difference in cost between the two filters, finite- and infinite-aperture, can be significant, which points to the desirability of using the infinite-aperture rho filter.