Deconvolution of Surface Seismic Data Using Vertical Seismic Profiles

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Abstract

Two major objectives of surface seismic data processing are collapsing the seismic source waveform into a spike and removing multiple events. The first objective, deconvolution, requires the shape of the source waveform. The second objective, de-reverberation, requires information about the position of the multiples. The source waveform's shape and the position of the multiples is not readily available from surface seismic data. A vertical seismic profile near the location of the seismic survey can provide an approximation of the source waveform and the location of the multiples. This information is available from vertical seismic profiles because dip filters are able to separate upgoing and downgoing waves. Once an algorithm extracts the source waveform and multiple information from the vertical seismic profile, processing algorithms can use the information and deconvolve the surface seismic section.

Introduction

Hubbard (1979) first published a method of obtaining the source waveform and the multiple information from vertical seismic profiles (VSPs) for deconvolving surface seismic data. I will extend Hubbard's method by first describing a model of the earth and defining the waveforms that exist within this model. Then, I will examine the methods of extracting the source waveform and multiple information from the VSP. Using this model, I will discuss the problem of using the information from the VSP for the entire surface seismic section. A multiple in this model is any seismic wave that reflected down at least once before reaching the surface.

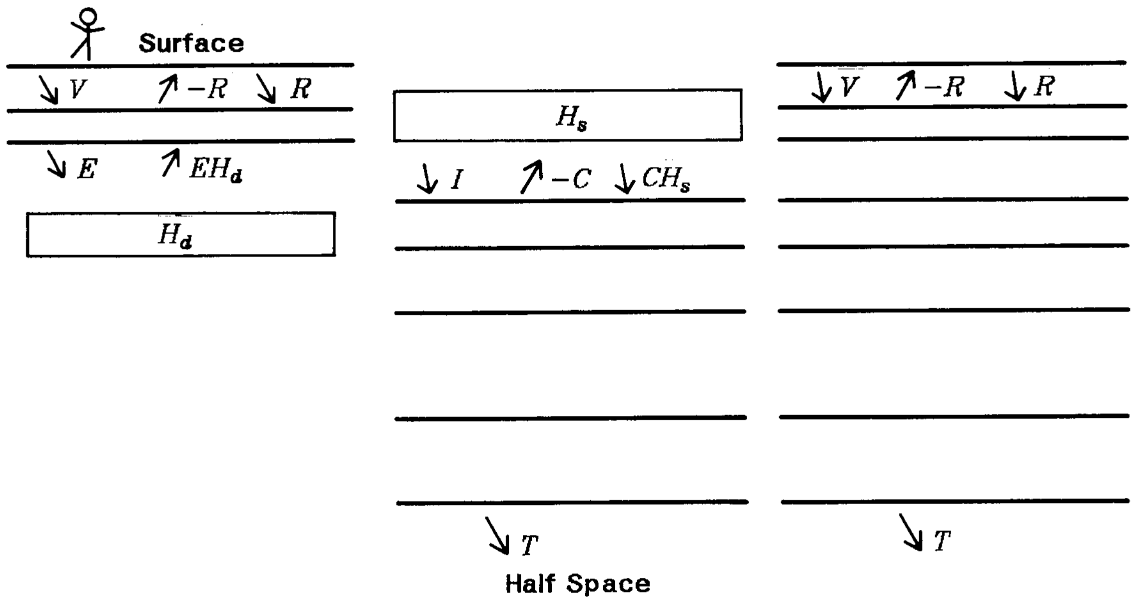

FIG. 1. The earth model consists of two sets of reflectors. The shallow reflectors are on the left. The deep reflectors are in the center. The model on the right combines the two sets of reflectors.

Definition of the Earth Model and Waveforms

First, consider a simple model of the earth and the waveforms in this model. Later, the model can become more complicated by adding features that allow the model to simulate the earth more closely. The first assumption of the model is that the earth consists of homogeneous elastic layers that are relatively constant over short distances. In other words, the subsurface can vary slowly as a function of horizontal distance. Variations may include slowly dipping layers or a thinning bed. The changes imply that the reflection characteristics will change slowly over the seismic section. In the subsurface neighboring the VSP, the subsurface is essentially constant with horizontal distance.

The second assumption is that all seismic waves arrive at normal incidence on the reflectors. The second assumption is valid for VSPs when the seismic source is close to the top of the well.

The third assumption is that the earth consists of two sets of reflectors, and that the two sets of reflectors are separable to form two different models. The first set of reflectors, the shallow reflectors, are shown on the left side of figure 1. The shallow reflectors include all reflectors from the surface of the ground down to the bottom of the weathering layer. For offshore surveys, the shallow reflectors include the seafloor and reflectors above competent rock. The second set of reflectors are the deep reflectors, which are shown in the center part of figure 1. The deep reflectors include all the reflectors starting from the bottom of the shallow reflectors and extending down to the basement. These two sets of reflectors each have a set of waveforms, which the following sections define below.

Waveforms of the Shallow Reflectors

Consider all the reflectors below the shallow reflectors as a "black box" with an impulse response of $H_d(Z)$. Note that the impulse response of the black box and all other waveforms will be expressed in the Z-domain. Multiplication of Z polynomials correspond to convolution in the time domain. If there was a geophone at four different locations in the shallow reflectors, the geophone would record four different waveforms.

1. $V(Z)$ is the waveform of the seismic source that is entering the ground. A geophone directly beneath the seismic source would record this waveform.
2. $E(Z)$ is the downgoing waveform that has passed through the shallow reflectors. If a geophone that recorded only downgoing waves were buried below the shallow reflectors, it would record the signal $E(Z)$.
3. $R(Z)$ is the upgoing waveform that a geophone at the surface would record.
4. $E(Z)\ H_d(Z)$ is the response of the deep reflectors when the waveform, $E(Z)$, is the input. If a geophone that only recorded upgoing waves was buried below the shallow reflectors, it would record the signal $E(Z)\ H_d(Z)$.

The equations that describe the above waveforms are given by

$$\begin{bmatrix} -E(Z)\ H_d(Z) \\ E(Z) \end{bmatrix} = \mathbf{M}_s \begin{bmatrix} -R(Z) \\ V(Z) + R(Z) \end{bmatrix} \tag{1}$$

where $\mathbf{M}_s$ is a 2x2 reflection matrix for the shallow reflectors (Claerbout, 1976). A reflection matrix, **M** , has the form

$$\mathbf{M} = \begin{bmatrix} F(Z) & Z^k\ G(Z^{-1}) \\ G(Z) & Z^k\ F(Z^{-1}) \end{bmatrix}$$

where $F(Z)$ and $G(Z)$ are both polynomials of Z, and k is the order of $F(Z)$ and $G(Z)$. The constant coefficients for the terms in $F(Z)$ and $G(Z)$ come from from formulas that use the reflection coefficients of the layers. If any two of the four above waveforms is provided, equation (1) can calculate the other two waveforms.

Separation of $E(Z)$ and $H_d(Z)$

Most VSP surveys have a geophone buried near the top of the well below the weathering layer, which corresponds to the bottom of the shallow reflectors. The buried geophone will record the sum of the downgoing and upgoing signals, or

$$G_b(Z) = E(Z) - E(Z)\,H_d(Z) \tag{2}$$

where $G_b(Z)$ is the signal recorded by the buried geophone. In order to be useful, the signal from the buried geophone has to be separated into its upgoing component, $E(Z)\,H_d(Z)$, and its downgoing component, $E(Z)$.

The first method of separating $E(Z)$ and $E(Z)H_d(Z)$ involves a simple window. Inspection of figure 1 shows that $E(Z)$ is the seismic waveform that enters the deep reflectors, and that $E(Z)\,H_d(Z)$ is the reflection back. In other words, $E(Z)$ has most of its energy near the beginning of $G_b(Z)$ because no signal comes from below until $E(Z)$ has entered the deep reflectors and reflected back. $E(Z)$ also has a much stronger amplitude then $E(Z)\,H_d(Z)$ because $E(Z)$ is the direct wave from the source. Since $E(Z)$ comes first and is stronger, a window of the first part of $G_b(Z)$ approximates $E(Z)$. An approximation for $E(Z)\,H_d(Z)$ comes from the rest of $G_b(Z)$.

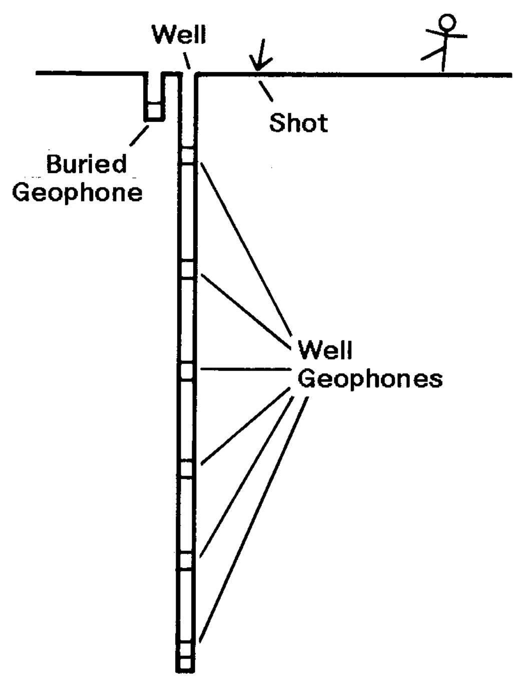

FIG. 2. The geometry of well geophones and the buried geophone during a vertical seismic profile.

The second separation method involves using the other geophone records from the well. Figure 2 shows that the separation between the buried geophone and the top geophone in the well is greater than the distance between the positions of geophones in the well. An interpolation algorithm can calculate the data between the top well geophone and the buried geophone. With the missing data, the buried geophone essentially becomes part of the rest of the VSP, which means a dip filter can separate the upgoing and downgoing waves in $G_b(Z)$. The downgoing wave is $E(Z)$, and the upgoing wave is $E(Z)\ H_d(Z)$.

Of the two methods outlined above, the window method is faster and will provide a good approximation of $E(Z)$ since most of the energy in $E(Z)$ is at the beginning. However, the window's algorithm's effectiveness depends on the accuracy of knowing where $E(Z)\ H_d(Z)$ starts in $G_b(Z)$. Any approximation of $E(Z)$ obtained through a window will contain energy from $E(Z)\ H_d(Z)$. The second method of interpolation and dip filtering depends on the accuracy of the missing data algorithm and the amount of noise. Since the top well geophone and the buried geophone are close to the surface, surface noise can dominant component of the signal. If there is a significant amount of surface noise, then the second method will not work. If the second method works, the algorithm will separate the energy in the downgoing wave from the energy in the upgoing wave.

Waveforms of the Deep Reflectors

The deep reflectors, which are in the middle part of figure 1, include all reflectors below the shallow reflectors. The shallow reflectors are a "black box" and have the impulse response $H_s(Z)$. The deep reflectors have four waveforms:

1. $I(Z)$ is the initial downgoing waveform that enters the deep reflectors.
2. $C(Z)$ is the upgoing waveform reflected back toward the shallow reflectors. If a geophone that only records upgoing waves was placed at the top of the deep reflectors, it would record $C(Z)$.
3. $C(Z)\ H_s(Z)$ is the downgoing waveform reflected back from the shallow reflectors when $C(Z)$ is the input. If a geophone that only records downgoing waves was placed at the top of the deep reflectors, it would record the sum of $I(Z)$ and $C(Z)\ H_s(Z)$.
4. $T(Z)$ is the downgoing waveform leaving the bottom of the deep reflectors. A geophone placed below the deep reflectors would record $T(Z)$.

The equations for this model are

$$\begin{bmatrix} 0 \\ T(Z) \end{bmatrix} = \mathbf{M}_d \begin{bmatrix} -C(Z) \\ I(Z) + C(Z)\ H_s(Z) \end{bmatrix} \tag{3}$$

where $\mathbf{M}_d$ is the reflection matrix for the deep reflectors. The zero on the left side of equation (3) implies that there is no signal is coming up from below the deep reflectors.

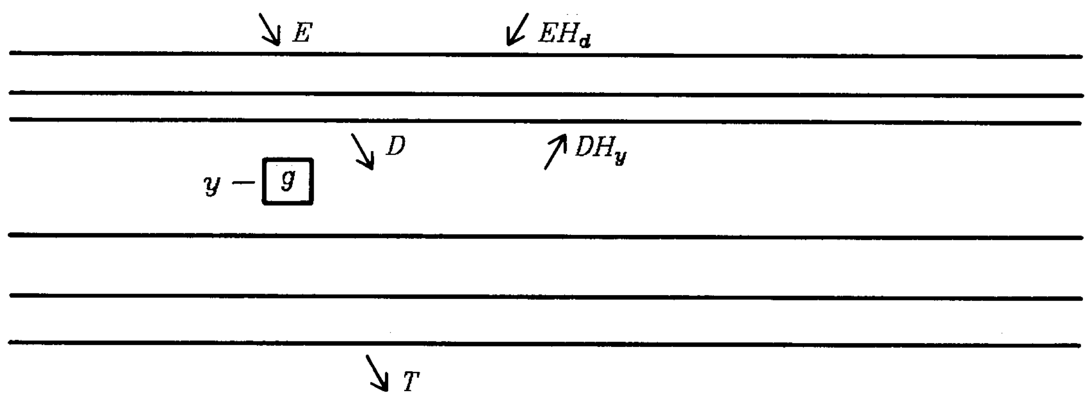

FIG. 3. The waveforms around the well geophone. The geophone is represented by the box at a depth y.

The Waveforms of the Well Geophone

The well geophone records waveforms at a depth y within the deep reflectors. Suppose the reflectors below the well geophone are a "black box". Their impulse response is $H_y(Z,y)$. The impulse response of the reflectors above the well geophone is $H'_y(Z,y)$. Figure 3 shows the two main waveforms located at the well geophone:

1. $D(Z,y)$ is the downgoing waveform at the depth y.
2. $D(Z,y)\, H_y(Z,y)$ is the upgoing waveform from the lower reflectors at a depth y

The equations for the well geophone waveforms are

$$\begin{bmatrix} 0 \\ T(Z) \end{bmatrix} = \mathbf{M}_y \begin{bmatrix} -D(Z,y)\, H_y(Z,y) \\ D(Z,y) \end{bmatrix} \tag{4a}$$

and

$$\begin{bmatrix} -D(Z,y)\, H_y(Z,y) \\ D(Z,y) \end{bmatrix} = \mathbf{M}'_y \begin{bmatrix} -E(Z)\, H_d(Z) \\ E(Z) \end{bmatrix} \tag{4b}$$

where $\mathbf{M}_y$ is the reflection matrix for the layers below the well geophone, and $\mathbf{M}'_y$ is the reflection matrix of the layers above the well geophone. The product of $\mathbf{M}_y$ and $\mathbf{M}'_y$ is the reflection matrix for the all the deep reflectors, or $\mathbf{M}_d$. Note that if the left hand side of equation (4b) was substituted into the right hand side of equation (4a), then the result is equation (3).

The well geophone at a depth y records the waveform

$$G_w(Z,y) = D(Z,y) - D(Z,y)\, H_y(Z,y)\,. \tag{5}$$

The downgoing waveform, $D(Z,y)$, contains all the multiple information that formed above the depth y. This fact may not be readily apparent, but examination of the seismic wave's path shows that $H_y(Z,y)$ does not contain any signal from multiples. The definition for multiples states that a multiple is any seismic wave that reflected down at least once. A well geophone at the depth below where the seismic wave reflected downward would record the multiple, as well as the direct wave. Therefore, $D(Z,y)$ contains all the multiples formed above the depth y. The waveform $H_y(Z,y)$ only contains the impulse response of the reflectors below the depth y. $H_y(Z,y)$ contains multiple information that formed below the depth y. Table 1 contains a list of the symbols for all the waveforms and impulse responses.

The Downgoing Waveform $D(Z,y)$

The downgoing waveform recorded by the well geophone, $D(Z,y)$, can be rewritten as

$$D(Z,y) = E(Z)\, A_1(Z,y) \tag{6}$$

where $A_1(Z,y)$ is the one way impulse response of the layers above the well geophone. In order to define the one way impulse response, suppose all the deep reflectors between the geophone and the shallow reflectors were isolated from the rest of the reflectors. If an impulse entered the top, a geophone at the bottom would record the waveform $A_1(Z,y)$. If the impulsive source and the geophone were switched, the geophone would record the same waveform, $k\, A_1(Z,y)$, owing to the principle of reciprocity. The only difference between the two waveforms is that the overall amplitude of the waveform is scaled by a constant k. Appendix A gives a derivation showing that the shape of the two waveforms remains the same, except for a amplitude scale factor of k.

Suppose the signal detected by the geophone went back up through the reflectors. Since the impulse responses of the reflectors are the same, then a geophone at the top would record the signal $[A_1(Z,y)\, A_1(Z,y)]$. $[A_1(Z,y)\, A_1(Z,y)]$ is the two way impulse response of the deep reflectors. In other words, a signal that goes down through the layers and then goes back has the two way impulse response

$$H'_y(Z,y) = A_1(Z,y)\, A_1(Z,y)\,. \tag{7}$$

Function	Terms	Meaning
Shallow Waveforms	$V(Z)$	Shot waveform
	$E(Z)$	Downgoing waveform leaving the bottom of the shallow reflectors
	$G_s(Z) = R(Z)$	Upgoing waveform that is recorded by a surface geophone
	$E(Z)\ H_d(Z)$	Upgoing waveform that is returning from the deep reflectors
	$G_b(Z) = E(Z) + E(Z)\ H_d(Z)$	Waveform recorded by the buried geophone
Deep Waveforms	$I(Z)$	Direct downgoing waveform coming from the shallow reflectors
	$C(Z)$	Upgoing waveform from the top of the deep reflectors
	$T(Z)$	Downgoing waveform leaving the bottom of the deep reflectors
	$C(Z)\ H_s(Z)$	Downgoing waveform that is reflected back from the shallow reflectors
Well Geophone Waveforms	$D(Z,y)$	Downgoing waveform at depth y in the deep reflectors
	$D(Z,y)\ H_y(Z,y)$	Upgoing waveform at depth y in the deep reflectors
	$G_w(Z,y) = D(Z,y) - D(Z,y)\ H_y(Z,y)$	Waveform recorded by the well geophone at depth y in the deep reflectors
Impulse Responses	$H_s(Z)$	Response of shallow reflectors to an upgoing impulse
	$H_d(Z)$	Response of deep reflectors to a downgoing impulse
	$H_y(Z,y)$	Response of reflectors below y to a downgoing impulse
	$H'_y(Z,y)$	Response of reflectors above y to an upgoing impulse
	$A_1(Z,y)$	One way response of the layers above y to an impulse
	$A_2(Z,y)$	Two way response of the layers above y to an impulse

TABLE 1. This table lists the waveforms. The table has four groups: waveforms in the shallow reflectors, waveforms in the deep reflectors, waveforms around the well geophone, and impulse responses of the reflectors.

The two way impulse response of the reflectors below a geophone at a depth of y is given by the term $H_y(Z,y)$. The impulse response of the entire set of deep reflectors, $H_d(Z)$, is given by the equation

$$H_d(Z) = H_y(Z,y)\ H'_y(Z,y) \tag{8}$$

This equation can be derived by supposing that the deep reflectors above y and the deep reflectors below y are two separate linear systems which are combined serially into one system. For two linear systems in serial order, the impulse response of the first system convolved with the impulse response of the second system yields the impulse response of the combined system. Therefore, $H_d(Z)$ is the product of $H_y(Z,y)$ and $H'_y(Z,y)$. If a geophone at a depth y can record both $H_y(Z,y)$ and $H'_y(Z,y)$, then equation (8) gives the impulse response of the deep reflectors.

Proposed Method of Deconvolving Surface Data

The five following steps define an algorithm for deconvolving surface seismic data.

1. Separate the upgoing waves and the downgoing waves on the VSP.

The ability to separate downgoing and upgoing waves is an important property of the VSP. The downgoing waves contain most of the information that is lost during regular surface seismic surveys. This first step obtains the downgoing waves, $D(Z,y)$, and the upgoing waves, $D(Z,y)\, H_y(Z,y)$, for each geophone position. Several algorithms exist that separate events that have different dips. For example, Seeman and Horowicz (1983) suggest modeling the upgoing and downgoing waves and using a least squares technique to separate them. Harlen (1983) has suggested an iterative approach of subtracting events based on their slope and amplitude.

2. Deconvolve the upgoing waves with the downgoing waves.

The second step obtains the impulse response of the layers below the well geophone. This step is represented by the equation

$$\frac{D(Z,y)\; H_y(Z,y)}{D(Z,y)} \;=\; H_y(Z,y)\;.$$

The numerator on the left side is the upgoing wave at the geophone, and the denominator is the downgoing wave. The result is the impulse response of the deep reflectors below the well geophone. The impulse response for a depth y helps identify primary events that are present on a surface seismic section at the depth y. The signal $H_y(Z,y)$ does not have any multiple events that were caused by the geology above y because the multiples were removed when the upgoing wave was divided by $D(Z,y)$. A surface-caused multiple that appears on a seismic section below a depth y will not show up in $H_y(Z,y)$ because the surface-caused multiples were removed when $D(Z,y)$ was removed.

3. Auto-convolve the downgoing wave, $D(Z,y)$, from the well geophone, and then deconvolve by the source waveform, $E(Z)$ from the buried geophone.

The third step produces the source waveform, $E(Z)$, convolved with the two-way impulse response of the layers above the geophone, $H'_y(Z,y)$. Combining equations (6) and (7) yields the equation

$$H'_y(Z,y) \;=\; \frac{D(Z,y)}{E(Z)}\;\frac{D(Z,y)}{E(Z)}$$

or

$$\frac{D^2(Z,y)}{E(Z)} \;=\; E(Z)\; H'_y(Z,y) \qquad (9)$$

The left side of equation (9) shows the result of the third step. The left hand term of equation (9) will be used as a deconvolution operator in the next step.

4. Deconvolve the surface data from around the well with the results of the third step.

This step deconvolves the surface data, yielding the impulse response of the earth. The signal measured by the surface geophone is given by

$$R(Z) \;=\; E(Z)\, H_d(Z)\, B_s(Z) + M(Z)\ . \tag{10}$$

$B_s(Z)$ is the one way impulse response of the shallow layers. Suppose the deep reflectors were not present and a geophone was placed below the shallow reflectors. The geophone would record the waveform $B_s(Z)$ if an impulse entered the top of the reflector. Since a seismic signal leaves from the surface and returns to the surface, the signal must travel through the shallow reflectors twice. Equation (10) should have two $B_s(Z)$ terms to represent the two passes. The other $B_s(Z)$ term is located in the term $E(Z)$, since $E(Z)$ is equal $V(Z)\, B_s(Z)$. The term $M(Z)$ corresponds to the seismic waves that just bounce around in the shallow reflectors and never enter the deep reflectors. In other words, $M(Z)$ is the two way impulse response of the shallow reflectors. The shallow reflectors are very thin compared to the deep reflectors, so any signal that enters them quickly passes through without producing a strong reflected signal. The main energy of $M(Z)$ is concentrated at the beginning of the record of the surface seismic trace. Therefore, $M(Z)$ in equation (10) can be removed, and equation (10) becomes

$$R(Z) \;=\; E(Z)\, H_d(Z)\, B_s(Z)\ . \tag{10a}$$

Deconvolving $R(Z)$ by the operator from the step 3 results in

$$R(Z)\, \frac{E(Z)}{D^2(Z,y)} \;=\; \frac{E(Z)\, H_d(Z)\, B_s(Z)}{E(Z)\, H'_y(Z,y)}$$

From equation (8), $H_d(Z)$ is

$$H_d(Z) \;=\; H_y(Z,y)\ \ H'_y(Z,y)$$

Combining the two above equations, equation (10a) becomes

$$R(Z)\, \frac{E(Z)}{D^2(Z,y)} \;=\; H_y(Z,y)\, B_s(Z) \tag{11}$$

The right hand side of equation (11) is the impulse response of the earth, which was the main objective.

5. Based on the results of step 4, design and apply a deconvolution operator from the third step to the rest of the surface data.

One assumptions of the model was that the subsurface geology around the well was fairly constant. Since the geology is fairly constant, the deconvolution operator from step 3 will work on traces from surface data near the well. As the geology starts the vary as a function of distance from the well, the operator from step three must change.

Judging the Deconvolution Operator

Any algorithm that adjusts the deconvolution operator from step 3 to matches the changing subsurface needs some feedback about the success of the operator. In other words, the algorithm needs the ability to determine whether an operator properly deconvolved a seismic trace. If the operator did not, then the algorithm must know how to alter the operator. This problem differs from other deconvolution problems owing to the amount of knowledge that is known. Instead of starting with little or no information about multiples and waveform shape, the algorithm starts out with an fairly accurate operator that includes multiple information and the source waveform shape from a very similar geology.

The simplest method of feedback to the algorithm is for a person to visually examine the trace after processing and make changes in the parameters used by the program. However, visual examination is not qualitative, and a high degree of experience would be needed to guide the computer to an accurate result. Therefore, the feedback process needs one or more parameters that the algorithm can compute by measuring changes in the data. If the deconvolution or de-reverberation is successful, these parameters will show it.

One example of a parameter is a measure of how gaussian the amplitude distribution is. In others words, do the amplitudes of the processed section come close to having a gaussian distribution. If a deconvolution is successful, primary events will have higher amplitudes and sharper waveforms. High amplitudes and sharp waveforms makes the amplitude distribution less gaussian. If the operator broadens the waveforms, the amplitude distribution becomes more gaussian. If the random background noise is stronger than the signal, the amplitude distribution becomes gaussian. The purpose of deconvolution is to keep the data from becoming more gaussian.

A second parameter that measures the success of a de-reverberation operation is the change in relative energy of known primary events. A successful de-reverberation operator moves the energy from multiple events to primary events. Therefore, the energy in a primary event increases relative to the energy in the multiple. If the program fails to increase the energy of the primary event, then the de-reverberation was unsuccessful.

These two parameters and any others which may present themselves define a space in which a deconvolution and de-reverberation operator exists. The purpose of this feedback is to guide the algorithm to a better operator. If the initial guess is close to the best operator, then convergence should happen quickly. Part of the research will involve how to make changes to the operator to reflect the values of the parameters.

Feasibility of This Method

The first two steps outlined above are common VSP processing routines used by industry. The last three steps will require more research and development. Several problems in these three steps are not resolved.

One major problem that has not been mentioned yet is noise. If the VSP has a significant amount of noise, then the deconvolution operator will not work on the surface section. Most VSPs have some type of noise from surface waves traveling down the hole. The noise is especially strong when the well geophone is near the top of the well. A preliminary processing step is attenuation of these surface waves on the VSP.

The second major problem is adapting the deconvolution operator as the subsurface geology changes and making a decision about whether the operator was successful. A deconvolution algorithm that looks good on paper does not always produce results that look good. Unless a seismic section looks 'good', a deconvolution algorithm does not work. What constitutes 'good' has to be defined numerically before building an algorithm that adapts deconvolution operators.

Two aspects of the above outline are innovative. Current methods of de-reverberation use only the signal from the well geophone to form operators. No one has published method that uses the information from the buried geophone with a similar earth model. The extra information from the buried geophone greatly helps deconvolve and de-reverberate seismic sections. The second aspect is the deconvolution of the rest of a section given that a few traces are correct. Past methods do not assumed that some traces are more correct than others. This method provides a new approach to this problem by specifying a different set of boundary conditions. Most present deconvolution programs make assumptions about the nature of the signal and what to expect at certain times. This method will introduce a lateral assumption that certain traces are more accurate than the others. This approach to deconvolution is the most promising aspect of using VSPs to deconvolve surface seismic sections, as it will have applications to seismic sections where no VSP data is available. If some traces on a section which were processed correctly, an algorithm can process the rest of the seismic section using the known correct traces.

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Appendix A

The following assumptions are made about the earth for this derivation:

1. The earth consists of horizontal elastic layers.
2. The travel time of a seismic wave across each layer is constant. The thickness of each layer is proportional to the layer's seismic velocity.
3. All seismic waves have normal incidence to the boundaries between the layers.
4. Each boundary has a reflection coefficient, c, and a transmission coefficient, t.
5. The layers are bounded on top and on bottom by a half space.
6. Multiplication by the delay operator, Z, corresponds to a delay needed for a seismic wave to cross a layer.

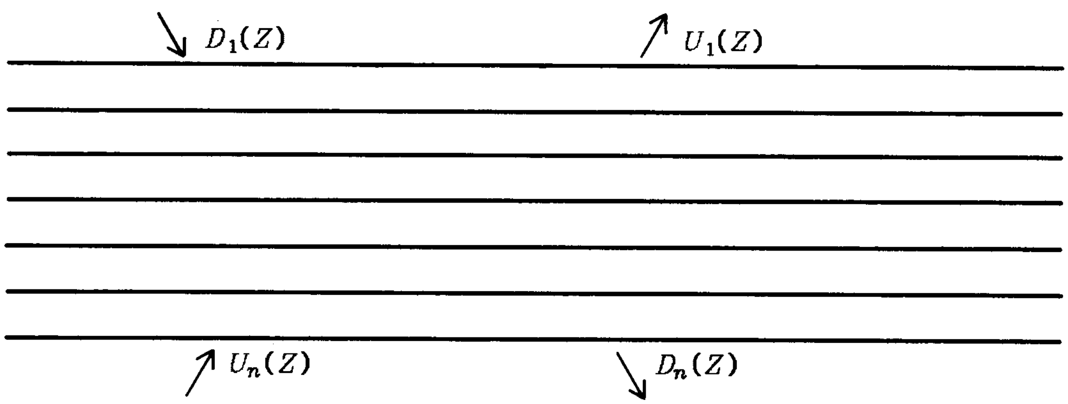

FIG. A1. A k-layered model of the earth. The four waves are incident on the layers from the two half spaces surrounding the layers.

Figure A1 shows the model of the earth and the four major waveforms. $D_1(z)$ is the input seismic waveform from the top of the layers. $U_1(Z)$ is the output seismic waveform at the top of the layers. $D_n(Z)$ is the output waveform from the bottom of the layers, and $U_n(Z)$ is the input waveform to the bottom of the layers. The equation for these four waveforms is given by the matrix:

$$\begin{bmatrix} U_n(Z) \\ D_n(Z) \end{bmatrix} = \frac{1}{T\sqrt{Z^k}} \begin{bmatrix} F(Z) & Z^k\, G(Z^{-1}) \\ G(Z) & Z^k\, F(Z^{-1}) \end{bmatrix} \begin{bmatrix} U_1(Z) \\ D_1(Z) \end{bmatrix} \qquad \text{(A1)}$$

where $F(Z)$ and $G(Z)$ are polynomials of Z and have an order k. The coefficients of $F(Z)$

and $G(Z)$ are functions of the reflection coefficients of the layers (see Claerbout, 1976). The term T is given by

$$T = \prod_{i=1}^{k} t_i$$

where t_i is the transmission coefficient of the ith layer when a seismic wave is traveling up. Therefore, T is the product of all the transmission coefficients of the layers.

Upward Impulse Response

Let $U_n(Z)$ be an impulse ($U_n(Z) = 1$), and assume there is not input from above ($D_n(Z) = 0$). Substituting these values into equation (A1), the equation becomes

$$\begin{bmatrix} 1 \\ D_n(Z) \end{bmatrix} = \frac{1}{T\sqrt{Z^k}} \begin{bmatrix} F(Z) & Z^k\, G(Z^{-1}) \\ G(Z) & Z^k\, F(Z^{-1}) \end{bmatrix} \begin{bmatrix} U_1(Z) \\ 0 \end{bmatrix} \qquad \text{(A2)}$$

Solving for $U_1(Z)$, equation (A2) becomes

$$1 = \frac{1}{T\sqrt{Z^k}} F(Z)\, U_1(Z)$$

or

$$U_1(Z) = \frac{T\sqrt{Z^k}}{F(Z)} \qquad \text{(A3)}$$

Therefore, $U_1(Z)$ equation (A3) is the upgoing impulse response of a series of layers when the impulse enters the layers from below.

Downward Impulse Response

The matrix in equation (A1) can be inverted to become

$$\begin{bmatrix} U_1(Z) \\ D_1(Z) \end{bmatrix} = \frac{T\sqrt{Z^k}}{H} \begin{bmatrix} Z^k\, F(Z^{-1}) & -Z^k\, G(Z^{-1}) \\ -G(Z) & F(Z) \end{bmatrix} \begin{bmatrix} U_n(Z) \\ D_n(Z) \end{bmatrix} \qquad \text{(A4)}$$

where

$$H = Z^k\, (F(Z)\, F(Z^{-1}) - G(Z)\, G(Z^{-1}))$$

Claerbout shows in his book that

$$F(Z)\, F(Z^{-1}) - G(Z)\, G(Z^{-1}) = \prod_{i=0}^{k} t_i \prod_{i=0}^{k} t'_i$$

where $\prod_{i=0}^{k} t'_i$ is the product of all the downward transmission coefficients. H then becomes

$$H = Z^k \prod_{i=0}^{k} t_i \prod_{i=0}^{k} t'_i$$

$$= Z^k \; T \; T' \tag{A5}$$

where

$$T' = \prod_{i=0}^{k} t'_i \; .$$

Substituting (A5) into equation (A4), (A4) becomes

$$\begin{bmatrix} U_1(Z) \\ D_1(Z) \end{bmatrix} = \frac{1}{T' \sqrt{Z^k}} \begin{bmatrix} Z^k \; F(Z^{-1}) & -Z^k \; G(Z^{-1}) \\ -G(Z) & F(Z) \end{bmatrix} \begin{bmatrix} U_n(Z) \\ D_n(Z) \end{bmatrix} \tag{A6}$$

Assume that $D_1(Z)$ is an impulse entering the layers from above, and that there is no input from below. Under these two assumptions, equation (A6) becomes

$$\begin{bmatrix} U_1(Z) \\ 1 \end{bmatrix} = \frac{1}{T' \sqrt{Z^k}} \begin{bmatrix} Z^k \; F(Z^{-1}) & -Z^k \; G(Z^{-1}) \\ -G(Z) & F(Z) \end{bmatrix} \begin{bmatrix} 0 \\ D_n(Z) \end{bmatrix}$$

Solving for $D_n(Z)$,

$$D_n(Z) = \frac{T' \sqrt{Z^k}}{F(Z)} \tag{A7}$$

Therefore, $D_n(Z)$ in equation (A7) is the downgoing impulse response of a series of layers when an impulse enters from the top. Equations (A3) and (A7) are identical waveforms except for a factor of a constant. This equality means that the transmitted impulse response of a sequence of layers is the same waveform, regardless of whether the impulse comes from the top or the bottom.