Proof that Every CPR is an Impedance

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If you look in FGDP or old versions of Lecture Notes Section 4.6, you will see that an impedance is *defined* to be a minimum phase filter (has a causal inverse). Instead, that should have been deduced. It should be deduced before Muir's rule number 3, that the sum of impedances is an impedance. Otherwise proof of rule 3 is either faked or very hard.

The word *filter* refers to a Fourier transform pair. Let f_t denote the time domain representation of the filter, and let $F(\omega)$ denote its Fourier representation. Two filters are inverse to one another if their Fourier transforms are inverse to one another. A filter is causal if f_t vanishes before $t=0$. A filter is said to be PR if the real part of its Fourier transform is positive. What should have been proven, and will be proven now is that if a filter is both causal and PR (called CPR) then its inverse filter is causal.

Impedance Defined From Reflectance

The size of the class of filters called *impedances* will be seen to be large because they are derived by transformation from an easily specified family of filters called *reflectances*, say c_t and its Fourier transform $C(\omega)$. To be a reflectance, the time function must be strictly causal and the frequency function must be strictly less than unity. By *strictly causal* it is meant that the time function vanishes both at zero time and before. For example, take $-1 < \rho < +1$ and the reflectance c_t to be an impulse of size ρ after a time Δt. The Fourier transform is

$$C \quad = \quad \rho Z \quad = \quad \rho \, e^{-i\omega\Delta t}$$

An impedance has been defined to be a causal filter with a causal inverse and with a Fourier transform whose real part is positive. From any reflectance C, it will be shown that the expression

$$R \;=\; \frac{1-C}{1+C}$$

generates an impedance. Because of the assumption that C has magnitude strictly less than unity, $C\bar{C} < 1$, there is never any problem with the denominator. It always has a numerator representation as a convergent series in positive powers of C. Since the reflectance C is causal, so are the powers which represent multiple bounces. This constructs a causal representation for R. The inverse of R is found by simply changing the sign of C. It too is always convergent and causal for the same reason. The last part of the proof that the expression $R = (1-C)/(1+C)$ always produces impedances from reflectances is to show that R has a real part that is positive. Multiply top and bottom by the complex conjugate.

$$\mathrm{Re}\, R \;=\; \mathrm{Re}\, \frac{(1-C)(1+\bar{C})}{positive} \;\overset{?}{\geq}\; 0$$

$$\mathrm{Re}\, R \;=\; \mathrm{Re}\, \frac{(1-C\bar{C}) \;+\; imaginary}{positive} \;\geq\; 0$$

The expression for $R(C)$ is easily back solved for $C(R)$, but the converse theorem, that every R generates an reflectance, is harder to show. But we intend to show it, along with a deeper theorem. A filter is said to be Positive Real (PR) if the real part of its Fourier transform is positive. A filter that is both causal and PR is called CPR. The deeper theorem is that every CPR has an inverse, hence is an impedance. This will be proven by showing that every CPR, say $\hat{R}$, can be used to construct a reflectance $\hat{C}$ which by being a reflectance implies that the CPR $\hat{R}$ is an impedance R. The back solving gives

$$\hat{C} \;=\; \frac{1-\hat{R}}{1+R}$$

Proof requires two things be shown. First, the magnitude of $\hat{C}$ must be less than unity. To show this, form the magnitude of the denominator and subtract that of the numerator. The result is four times the real part of $\hat{R}$ which is positive. Second, $\hat{C}$ must be proven causal. This is much harder to prove. The denominator $1+\hat{R}$ can be expanded into a sum of positive powers of R hence of positive powers the delay operator. But the convergence of the series is not assured because nothing requires $\hat{R}$ to be less than unity.

Before continuing with this proof, an intermediate theorem is needed, namely that a scaled impedance is another impedance.

A Scaled Impedance is an Impedance

Let $\alpha>0$ be a real, positive scaling constant. It will be shown that $R' = \alpha R$ is another impedance. The method is by constructing a reflectance C' which will produce R'. An implicit definition of C' is:

$$\frac{1 - C'}{1 + C'} = \alpha \frac{1 - C}{1 + C}$$

Cross multiply

$$(1 - C')(1 + C) = \alpha (1 - C)(1 + C')$$

Bring terms depending on C' to the right.

$$(1-\alpha) + C(1+\alpha) = [(\alpha+1) + C(1-\alpha)]\, C'$$

Define

$$\rho = \frac{1 - \alpha}{1 + \alpha} \qquad \textit{where} \qquad -1 < \rho < 1$$

Solving for C' gives an explicit definition

$$C' = \frac{C + \rho}{1 + \rho C}$$

For C' to be a reflectance, two things must be verified. It must be causal, and it must have a spectrum strictly less than unity. The causality requires that the denominator be expandable into a convergent numerator, and it is. For unit boundedness, observe that the magnitude of the denominator minus that of the numerator is positive.

$$(1 + \rho C)(1 + \rho \bar{C}) - (C + \rho)(\bar{C} + \rho) \overset{?}{>} 0$$

$$(1 - C\bar{C})(1 - \rho^2) > 0$$

Since C' is proven to be a reflectance, the associated $R'=\alpha R$ is proven to be an impedance.

Every Causal-Positive-Real (CPR) is an Impedance

Now the proof can be completed that every causal PR filter has a causal inverse. The unresolved question was whether

$$\hat{C} = \frac{1 - \hat{R}}{1 + R}$$

can be proven causal. Consider first another function.

$$B = \frac{1 - \varepsilon \hat{R}}{1 + \varepsilon R}$$

Choose ε small enough that for all ω, $\varepsilon |\hat{R}| < 1$. This ensures a convergent expansion for the denominator in terms of positive powers of R, which contains only positive powers in the delay operator. Thus B is a reflectance and its corresponding impedance is $\varepsilon \hat{R}$. But an impedance can always be scaled by a positive number. Taking the number to be $1/\varepsilon$ shows that $\hat{R}$ is an impedance. This completes the proof that every CPR is an impedance.

Impedances arise more easily than we thought. It is not necessary to have a reflectance C to insert into the relation $R = (1-C)/(1+C)$. We only need to have a CPR. They can be constructed in many ways. The sum of two CPRs is a CPR because summing does not destroy causality, nor does it destroy positivity. It was also shown that CPRs are impedances, so they are always invertable and CPRs being impedances can be scaled by a positive constant. These three ways of combining CPRs to get other CPRs are called Muir's rules.

Reflectance of a Sum of Impedances

Before I had proven that every CPR is an impedance, it was really tough to prove that impedances sum to other impedances. But the intermediate stages of the proof are interesting, so it is preserved below.

$$R_1 + R_2 = \frac{1 - C_1}{1 + C_1} + \frac{1 - C_2}{1 + C_2} = \frac{1 - C_1 C_2}{1 + C_1 + C_2 + C_1 C_2}$$

$$R_1 + R_2 = \frac{1 - \dfrac{C_1 + C_2 + 2C_1 C_2}{2 + C_1 + C_2}}{1 + \dfrac{C_1 + C_2 + 2C_1 C_2}{2 + C_1 + C_2}}$$

What must be proven is that C' is a reflectance, where

$$C' = \frac{C_1 + C_2 + 2C_1 C_2}{2 + C_1 + C_2} = \frac{C_1(1 + C_2) + C_2(1 + C_1)}{(1 + C_1) + (1 + C_2)}$$

To reduce algebraic verbosity use the temporary substitutions $a = 1 + C_1$ and $b = 1 + C_2$.

$$C' = \frac{(a-1)b + (b-1)a}{a+b} = \frac{-(a+b) + 2ab}{a+b}$$

Form the magnitude of the denominator and subtract that of the numerator.

$$(a+b)(\bar{a}+\bar{b}) - [(a+b)(\bar{a}+\bar{b}) - 2ab(\bar{a}+\bar{b}) - 2(a+b)\bar{a}\bar{b} + 4ab\bar{a}\bar{b}] \overset{?}{>} 0$$

$$ab(\bar{a}+\bar{b}) + (a+b)\bar{a}\bar{b} - 2ab\bar{a}\bar{b} \overset{?}{>} 0$$

$$a\bar{a}(b + \bar{b} - b\bar{b}) + b\bar{b}(a + \bar{a} - a\bar{a}) \overset{?}{>} 0$$

The two terms are symmetric in a and b so it suffices to examine either one.

$$a + \bar{a} - a\bar{a} \overset{?}{>} 0$$

$$(1-C) + (1-\bar{C}) - (1-\bar{C})(1-C) \overset{?}{>} 0$$

$$1 - C\bar{C} > 0$$

The result that two impedances can be added to get another impedances has been shown by constructing the required reflectance.

History

Founded 1885
Opened 1891

PRESIDENTS

David Starr Jordan, 1891-1913
John Casper Branner, 1913-15
Ray Lyman Wilbur, 1916-43
Donald B. Tresidder, 1943-48
J. E. Wallace Sterling, 1949-68
Kenneth S. Pitzer, 1968-70
Richard W. Lyman, 1970-80
Donald Kennedy, 1980-

SCHOOLS & INSTITUTES*

Law—1908
Medicine—1909
Education—1917
Hoover Institution—1919
Food Research Institute—1921
Graduate School of Business—1925
Engineering—1925
Humanities & Sciences—1949
Earth Sciences—1963

*Date signifies establishment of the present school or institute; most existed earlier as programs or departments. School of Earth Sciences succeeds School of Mineral Sciences, established 1947.

Students

ADMISSIONS

Freshman

	Men	Women	Total
Applicants	8,292	5,737	14,029
Enrollees	870	679	1,549
Transfers (Autumn quarter only)			
Applicants	1,153	802	1,955
Enrollees	100	66	166

Deadline for completed applications for freshman admission: January 1, 1982.

Deadline for completed applications for autumn quarter transfer admission: April 1, 1982.

Group Information Sessions: Monday through Friday 9:30, 3:15; Saturday 9:30 (October through December).

For additional information regarding application procedure, admission requirements, etc., write Fred A. Hargadon, Dean of Admissions, Stanford, California 94305.

GRADUATE & PROFESSIONAL PROGRAMS

	Applied	Enrolled
Business	5,454	337
Earth Sciences	268	58
Education	411	110
Engineering	2,658	665
Humanities & Sciences	3,676	441
Law	4,011	176
Medicine	6,206	132

4.5 Stretching Tricks

Fourier analysis is generally inappropriate for time or space variable operations. This is unfortunate because other methods are often more costly than methods based on the fast Fourier transform program. Stretching tricks often enable Fourier methods to succeed where they otherwise would fail. The most obvious application (rare in practice) is to sample the time axis of the seismograms more coarsely at late times. This is reasonable because the earth Q has dissipated late high-frequencies. And it saves computer memory. After resampling, the spectrum of the data is more time invariant, so time invariant filters are more appropriate. Anyway, time invariant filters after resampling may be applied with Fourier transforms, whereas before resampling, time variable filters could not be applied with Fourier transforms.

Stolt Stretch

A more important application of time axis stretching was devised by Bob Stolt shortly after he developed his Fourier transform migration method. The great strength and the great weakness of the Stolt migration method is that it uses Fourier transformation over depth. This is a strength because it makes his method much faster than all other methods. And it is a weakness because it requires a velocity which is a constant function of depth. The earth velocity typically ranges over a factor of two within the seismic section, and the effect of velocity on migration tends to go as its square. To ameliorate this difficulty, Stolt suggested stretching the time axis to make the data look more as though it had come from a constant velocity earth. Stolt proposed the stretching function

$$\tau(t) \quad = \quad \left[\frac{2}{v_0^2} \int_0^t \ t \ v_{RMS}^2(t) \ dt \right]^{1/2} \tag{1a}$$

where

$$v_{RMS}^2(t) \quad = \quad \frac{1}{t} \int_0^t v^2(t) \ dt \tag{1b}$$

At late times, which are associated with high velocities, Stolt's stretch implies that τ grows faster than t. The τ-axis will be uniformly sampled to allow the fast Fourier transform.

Thus at late time the samples are increasingly dense on the t-axis. This is the opposite of what earth Q and the sampling theorem suggest, but most people consider this a fair price.

The most straightforward derivation of (1) is based on the idea of matching the curvature of ideal hyperbola tops to the curvature on the stretched data. The equation of an ideal hyperbola in (x,τ)-space is

$$v_0^2\, \tau^2 \quad = \quad x^2 + z^2 \tag{2}$$

Simple differentiation shows that the curvature at the hyperbola top is

$$\left.\frac{d^2\tau}{dx^2}\right|_{x=0} \quad = \quad \frac{1}{\tau\, v_0^2} \tag{3}$$

It may be shown that in a stratified medium the same relation holds except that the velocity is replaced by the RMS velocity.

$$\left.\frac{d^2 t}{dx^2}\right|_{x=0} \quad = \quad \frac{1}{t\ v_{RMS}^2} \tag{4}$$

We seek a stretched time $\tau(t)$ on the stratified medium that effectively replaces equation (4) with equation (3). We would like to match the x-dependent curves for all x. But that would overdetermine the problem. Instead we could just match the derivatives at the hyperbola top, *i.e.* the second derivative of $\tau[t(x)]$ with respect to x at $x=0$. With the substitutions (3) and (4), this gives an expression for $\tau\, d\tau / dt$ which after integrating and square root yields (1).

A different derivation of the stretch will give a more accurate result at steeper angles. Instead of matching hyperbola curvatures at the tops, the hyperbola slopes are matched at some distance out on the flank. It is the flanks of the hyperbola which actually migrate, not the tops, so this result will be more accurate. Algebraically, it is also an easier derivation, because only *first* derivatives are needed. Differentiating equation (2) with respect to x gives

$$\frac{d\tau}{dx} \quad = \quad \frac{x}{\tau\, v_0^2} \tag{5}$$

There is an analogous expression in stratified media. To obtain it, solve $x = \int v \sin\theta dt = p \int v^2 dt$ for $p = dt / dx$ getting

$$\frac{dt}{dx} \quad = \quad \frac{x}{\int_0^t v^2(p,t)\ dt} \tag{6}$$

Expressions (5) and (6) will play the same role as (3) and (4), but (5) and (6) are valid everywhere, not just at the hyperbola top. Differentiating $\tau(t)$ gives

$$\frac{d\tau}{dx} = \frac{d\tau}{dt}\frac{dt}{dx} \tag{7}$$

Inserting (5) and (6) into (7) gives

$$\frac{x}{\tau v_0^2} = \frac{d\tau}{dt}\frac{x}{\int_0^t v^2(p,t)\,dt} \tag{8}$$

$$\tau\, d\tau = \left[\frac{1}{v_0^2}\int_0^t v^2(p,t')\,dt'\right] dt \tag{9}$$

Integrating (9) gives $\tau^2/2$ on the left. Then taking the square root gives (1a) with a new definition for RMS velocity.

$$v_{RMS}^2(t) = \frac{1}{t}\int_0^t v^2(p,t)\,dt \tag{1c}$$

The thing which is new is the presence of the Snell parameter p. In a stratified medium characterized by some velocity, say $v'(z)$, the velocity $v(p,t)$ is defined for the tip of the ray that left the surface at an angle with a stepout p. In practice, what value of p should be used? The best procedure is to look at the data and measure the $p = dt/dx$ of those events which you wish to migrate well. A default value is $p = 2(sin 30^\circ)/(2.5$ km/sec) = .4 millisec/meter. The factor of 2 is from the exploding-reflector model.

Gazdag's V(x) Method

The phase-shift method of migration is attractive because it allows for arbitrary depth variation in velocity and arbitrary angles of propagation up to 90. Unfortunately lateral variation in velocity is not permitted because of the Fourier transformation over the x-axis. To ameliorate this difficulty, Jeno Gazdag proposed an interpolation method. Recall from Section 1.3 that the phase-shift method 2-D Fourier transforms the data $p(x,t)$ to $P(k_x,\omega)$. Then $P(k_x,\omega)$ is downward continued in steps of depth by multiplication with $\exp[ik_z(\omega,k_x)\Delta z]$. Gazdag proposes several reference velocities, say v_1, v_2, v_3, and v_4. He downward continues one depth step with each of the velocities, obtaining several reference copies of the downward continued data, say P_1, P_2, P_3, and P_4. Then each of the P_j is inverse Fourier transformed over k_x to $p_j(x,\omega)$. Then at each x, the reference waves of nearest velocity are interpolated to give a final value, say $p(x,\omega)$ which is retransformed to $P(k_x,\omega)$ ready for another step. This appears to be an inefficient method, duplicating the usual migration computation for each velocity. Surprisingly, the method seems to be successful, perhaps because of the peculiar nature of computation using an array processor.