Sources for Finite Difference Forward Modeling

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Abstract

Wave propagation simulations using numerical methods require some source condition. A method using Hankel transforms for the generation of "point" sources on finite difference meshes is presented. These point sources may be activated at any time and superimposed into complex source arrays giving some of the flexibility required for simulations of real situations

Introduction

Numerical simulations of wave propagation using finite differences require some initial or source conditions. The most obvious way is to begin with a displacement field of zero and to apply some excitation force which corresponds to the seismic source. The excitation force is time dependent and is turned on at some time t=0. Although it may be easy to do in a continuum, special care must be taken in the discrete case to avoid numerical errors. An easier method is to solve the wave equation analytically at some time $t > 0$ and superimpose this solution on the finite difference mesh, thereafter allowing the numerical scheme to take over the wave propagation. This is most easily done by convolving some given source with a wavefront impulse response function.

If the region surrounding the source is homogeneous the wavefront is a circle, in 2D, of known amplitude. Usually the source waveform is known in 1D as a function of time in the far field so the circular wavefront should be of sufficiently large radius to satisfy this criterion for the given source waveform. However, such a large source region is wasteful in terms of model space and therefore overall computation time. It is possible to propagate such circular wavefronts backward in time to the source time thereby decreasing the size of the source zone to the size of the waveform. A fast way to do this is by phase shift methods in the frequency domain. Because of the circular symmetry of the wavefront, 2D Fourier transforming is identical with Hankel transformation.

Modeling Scheme

Although the finite difference scheme is independent of the source generation I will briefly outline the technique for completeness. The acoustic wave equation is given by

$$\partial_{tt} W = KLW = v^2 \rho L W \qquad (1)$$

W denotes the pressure wavefield, K denotes the bulk modulus , v is velocity, ρ is density and L is a differential operator defined below.

$$L = \partial_x \frac{1}{\rho} \partial_x + \partial_z \frac{1}{\rho} \partial_z = L_x + L_z \qquad (2)$$

The values of $L_x W$ and $L_z W$ are computed using Fourier derivatives so the spatial derivatives are exact to the Nyquist spatial frequencies. Specifically these values are computed using

$$L_u W = F^{-1}\{ik_u F\{\frac{1}{\rho} F^{-1}\{ik_u F\{W\}\}\}\} \quad , \quad u = x , z \qquad (3)$$

Finite differencing over time using central differences gives the following explicit formula for the wavefield at a future time step in terms of the previous two time levels.

$$W_{x,z}^{t+1} = \Delta t^2 v^2 \rho (LW)_{x,z}^t + 2 W_{x,z}^t - W_{x,z}^{t-1}$$

Stability is analyzed by substituting a complex exponential trial solution $\exp[i(k_x + k_z - wt)]$ into equation (1).

$$\frac{e^{iw\Delta t} + e^{-iw\Delta t} - 2}{\Delta t^2} = \frac{-K}{\rho}(k_x^2 + k_z^2)$$

$$\rightarrow \quad \frac{2}{\Delta t^2}(\cos(w\Delta t) - 1) = -v^2 k^2$$

$$\rightarrow \quad \sin^2(w\Delta t / 2) = (vk\Delta t)^2 / 2$$

For real w, that is there is no real exponential part to the solution, it is required that $sin(w\Delta t / 2) \leq 1$ so the stability criterion is $v\Delta t / \Delta h \leq \sqrt{2} / \pi$. Here the value $\Delta h = min(\Delta x, \Delta z)$ so the maximum Nyquist wavenumber k is $\pi / \Delta h$.

Source generation

Effective point sources are generated by initializing W for two time steps at $t = -\Delta t$ and $t = 0$ in such a way as to make a compact source region which will generate the specified far field source $s(t)$. The wavefield is initialized at some time $t_0 > T$ where T is the length of the source wavelet in time units. This is done by applying equation (4).

$$W_{x,z} = s(r/v - t_0) = s(\sqrt{(x-x_0)^2 + (z-z_0)^2}/v + t_0) \qquad (4)$$

The source has been centered at (x_0,z_0). The wavefield is Fourier transformed and a phase shift is applied to "explode" the circular wavefront into an inward and outward propagating components. The method is demonstrated using the one dimensional example of Figure 1 and equation (5).

$$F\{\delta(x+X)+\delta(x-X)\} = e^{-ik_xX} + e^{ik_xX} = 2\cos(k_xX)F\{\delta(x)\} \qquad (5)$$

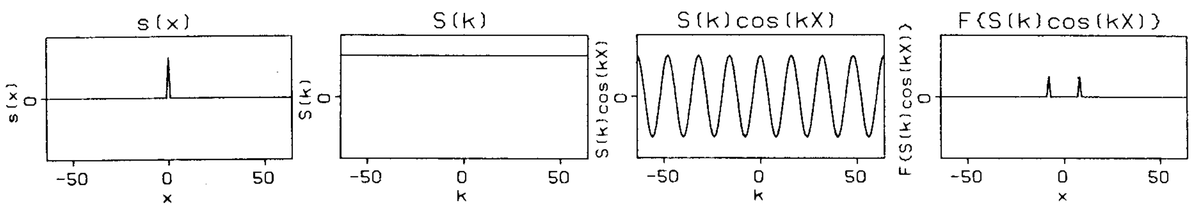

FIG. 1. A one dimensional example illustrating how source explosion corresponds to multiplication of the Fourier transform by a cosine.

From the one dimensional example it is clear that in 2D the appropriate phase shift is obtained by replacing x like terms with r like terms. Therefore, the phase shifted wavefield is given by

$$W_{k_x,k_z} = constant \ \cos(kR)F\{W_{x,z}\}$$

The value R is simply the required propagation radius and is set to $v(t_0 - \Delta t)$ and vt_0 for generation of the wavefield at $t = -\Delta t$ and $t = 0$ respectively and k is the wavenumber given by $\sqrt{k_x^2+k_z^2}$. The resultant wavefield has an unwanted circular wavefront at radius $R+vt_0$ and this can be removed by multiplying by some circular clipping function such as a

$\pi(r/(2vt_0))$ or some smoother function. Figures 2(a) through 2(f) illustrate each step of the method for a second derivative Gaussian source.

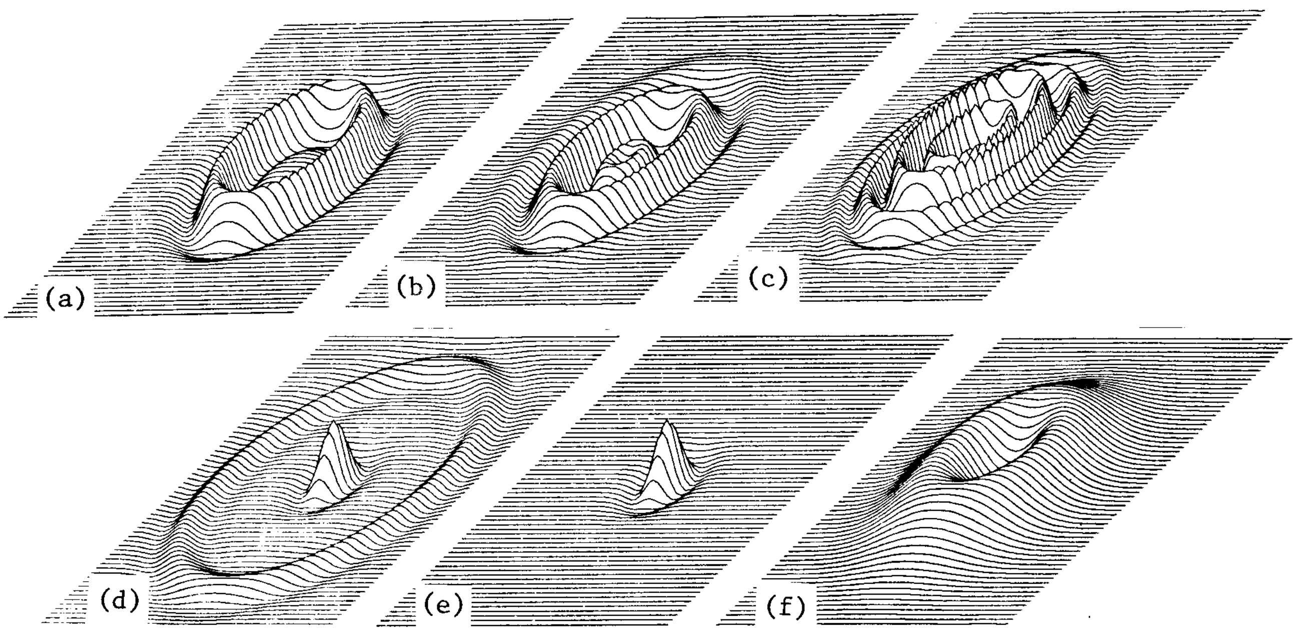

FIG. 2. The steps involved to generate a compact source. (a) Generation of a circular wavefront with the appropriate waveform. (b) Fourier transformation. (c) Cosine multiplication. (d) Inverse transformation. (e) Removal of the unwanted part. (f) Fourier transformation to show the resulting source spectrum.

Discussion

A quicker version of the above method is to simply rotate the one dimensional Fourier transformed source $s(w)$ about the origin of two dimensional (k_x,k_z) plane to generate a circularly symmetric source at $t = 0$. The $t = -\Delta t$ wavefield would be obtained by rotating $e^{iw\Delta t}s(w)$ about the origin in a similar fashion. The only drawback here is that there may be interpolation artifacts if one interpolates from the sampled $s(w)$ function to the (k_x,k_z) plane unless care is taken in the interpolation and in specifying a well behaved smooth $s(w)$. Of course this would not be a problem if $s(w)$ is known analytically. In general however, I expect the original method outlined to be the more robust of the two.

Examples

Two examples are shown in Figures 3 and 4 to demonstrate the flexibility of the method. Both illustrations contain pictures of the wavefield as time evolves for some source distribution. Figure 3 shows how a wavefront evolves resulting from three nearby point sources which are activated simultaneously. Figure 4 is an example of a line source of limited extent. Notice the edge effects. In both these examples I used a first derivative Gaussian for the source $s(t)$. Observe the waveform is no longer a first derivative Gaussian in linear parts of the wavefront in Figure 4. This is related to superposition effects and can be considered a result of the one dimensionality of the linear source.

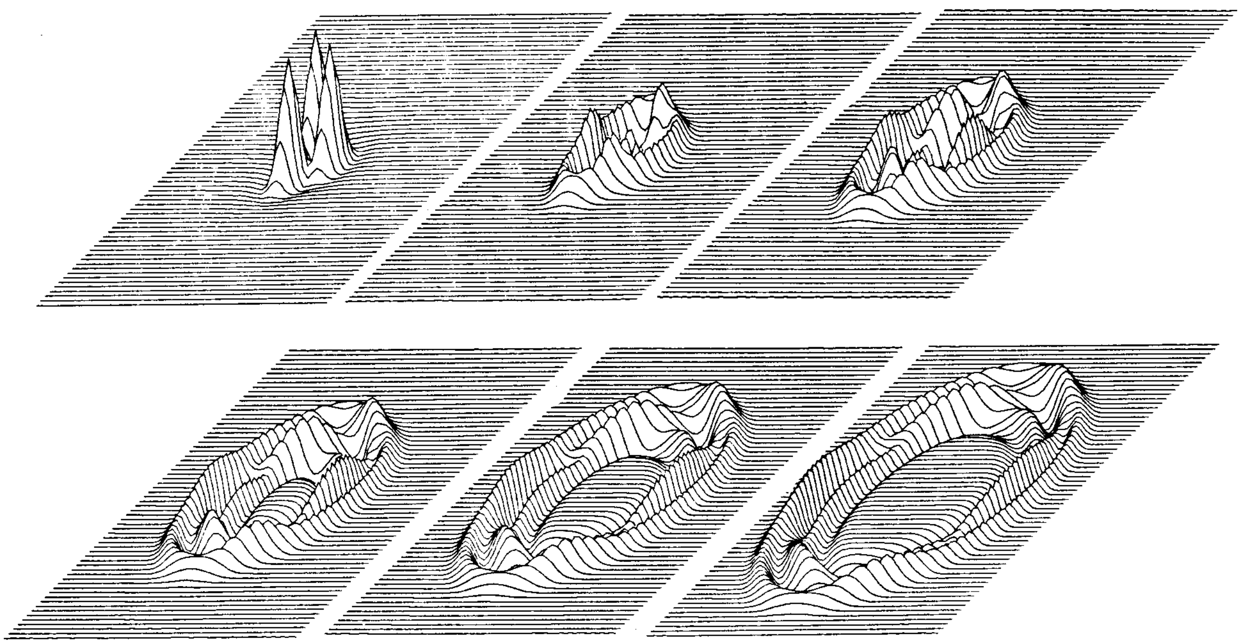

FIG. 3. Evolution of a wavefield resulting from three nearby first derivative Gaussian point sources activated simultaneously.

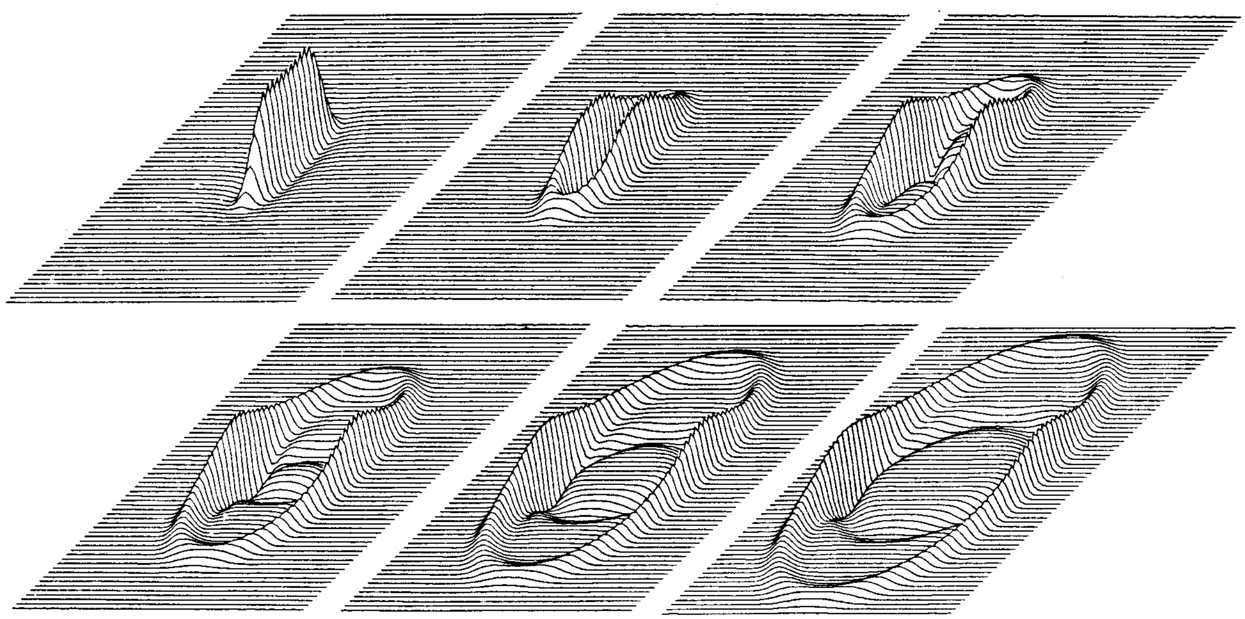

FIG. 4. Evolution of a wavefield resulting from a linear source of limited extent. The linear source is made of several aligned first derivative Gaussian point sources.

Conclusions

The Hankel transform technique for generating point sources on finite difference meshes is a flexible fast way to introduce some arbitrary source distribution in some limited homogeneous portion of the model space.

Acknowledgments

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