Slant Stack and Velocity Stack Inverse Filtering

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Introduction: Missing data and nonunique inversions

Any attempt at performing an inversion that is inherently nonunique must make some strong assumptions about the data that is missing. If we are fortunate to have enough data on hand to guarantee a unique inverse, or when noise is present, a unique least squares inverse, the addition of more data may make a small impact on the problem. This situation degenerates when the available input data becomes sparse and as the inverse (or as the case may be, the least squares inverse) becomes ill-conditioned. Various general methods exist that substitute other operators for the inverse if it is ill-conditioned or nonexistent. Among them are stochastic inverses (those that weight the diagonal of the forward operator before inversion) and generalized inverses, or pseudoinverses (Strang, sec. 3.4). Each of these methods makes an assumption about the missing data. The missing data may be estimated by first performing a pseudoinversion of the data, followed by applying the forward operator that extrapolates the missing data points. This paper will concentrate on determining the pseudoinverses of two commonly used operators in seismic processing: slant stacking and normal moveout stacking. Data that is fed into these processes, for example common midpoint or common shot gathers, usually have the property that they are sparsely sampled in the offset dimension h and bounded by a narrow range of offsets. The first limitation gives rise to aliasing artifacts, while the second, the spatial limitation, gives rise to truncation artifacts. This paper will also be concerned with the effects of truncation, or the lack of wide-offset data, on the quality of the inversion.

Projection operators and data truncation

Consider a data set **d** that is the result of a known linear process **L** operating on some unknown **u**. In this and the following sections, bold-face lower case letters ("**u**") will refer to either discrete vectors or functions on a continuous domain, while bold-face upper case letters ("**L**") will refer to matrices or continuous linear operators. If the data is incompletely sampled, let the unsampled portion of the data be set to zero. This is equivalent to post-multiplying **L** with a projection operator **P** that incorporates truncation and subsampling, as well as irregular sampling. For good measure, let the data be corrupted with a bit of additive noise **n**.

$$\mathbf{d} = \mathbf{PLu} + \mathbf{n} \tag{1}$$

$$P_{ij} \ (elements\ of\ \mathbf{P}) \equiv \delta_{ij} \begin{cases} 1 & (present\ data) \\ 0 & (missing\ data) \end{cases} \tag{2}$$

The projector **P** can be considered to be either a square matrix with 1's or 0's down the diagonal, or a non-square matrix with a single 1 somewhere in each row. The operator **L** may be nonsingular and invertible, so that if "all" the data were present, a unique and exact **u** can be obtained. Even in the presence of noise **n**, a unique estimate **u** can still be recovered by least squares. With the presence of the projector **P**, a null space may be introduced into the linear system **PL**. This does not say that **PL** always becomes singular, but in the case of slant stacking described in the next section, it will be seen that a nontrivial null space always arises from truncation of the data. The simplest case of degeneration to singularity is where **P** leaves fewer data values than unknowns. The solution will certainly be nonunique when it exists; the least squares solution is also singular. The least squares version of system (1) is

$$\mathbf{L}^T\mathbf{PLu} = \mathbf{L}^T\mathbf{d}. \tag{3}$$

Assuming that **L** has an inverse, and likewise $\mathbf{L}^T$, the null space of $\mathbf{L}^T\mathbf{PL}$ is equivalent to the null space of **PL**. Therefore the least squares system has the same null space as system (1).

There are two ways to overcome the non-uniqueness of equation (3) when the least squares operator is singular. The first is to apply the pseudoinverse $\mathbf{L}^+$ of **L** to **d**. The second is to positively weight the diagonal of $\mathbf{L}^T\mathbf{PL}$ until the modified linear system $\mathbf{L}^T\mathbf{PL} + \mathbf{D}$ becomes nonsingular. The pseudoinverse solution $\mathbf{u} = \mathbf{L}^+\mathbf{d}$ guarantees that **u** is identically zero in the null subspace of **PL**, while the second alternative may allow nonzero values of **u** in the null space, since the operator itself has been perturbed so that it is no longer singular. In the following sections the pseudoinverse solution will be detailed for two operators that are important in seismic processing: slant stacking and velocity stacking.

Pseudoinverses

Also called generalized inverses, pseudoinverses are most easily characterized by their singular value decomposition (SVD). Consider the operator **PL** of the last section. Let the unique SVD of **PL** be:

$$\mathbf{PL} \;=\; \begin{bmatrix} \mathbf{U}_p & \mathbf{U}_o \end{bmatrix} \begin{bmatrix} \Lambda_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_p^T \\ \mathbf{V}_o^T \end{bmatrix} \tag{4}$$

where $\mathbf{U}_p$, $\mathbf{U}_o$ and $\mathbf{V}_p$, $\mathbf{V}_o$ are partitions, respectively, of two unitary matrices **U** and **V**. The matrix Λ_p consisting of the singular values is diagonal. Assume **P** is square here, so that $\mathbf{PP} = \mathbf{P}$ and $\mathbf{P}^T = \mathbf{P}$. Consequently,

$$\mathbf{L}^T\mathbf{P} \;=\; \begin{bmatrix} \mathbf{V}_p & \mathbf{V}_o \end{bmatrix} \begin{bmatrix} \Lambda_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_p^T \\ \mathbf{U}_o^T \end{bmatrix} \tag{5}$$

Because $\mathbf{U}^T\mathbf{U} = \mathbf{I}$, the least squares functional shares a similar SVD:

$$\mathbf{L}^T\mathbf{PL} \;=\; \begin{bmatrix} \mathbf{V}_p & \mathbf{V}_o \end{bmatrix} \begin{bmatrix} \Lambda_p^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_p^T \\ \mathbf{V}_o^T \end{bmatrix} \tag{6}$$

Equation (6) also happens to be the eigenvalue decomposition of the least squares system $\mathbf{L}^T\mathbf{PL}$. By definition, the pseudoinverse of **L** is formed by inverting the nonzero singular values comprising the diagonal matrix Λ.

$$(\mathbf{PL})^+ \;\equiv\; \begin{bmatrix} \mathbf{V}_p & \mathbf{V}_o \end{bmatrix} \begin{bmatrix} \Lambda_p^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_p^T \\ \mathbf{U}_o^T \end{bmatrix}. \tag{7}$$

The pseudoinverse of $\mathbf{L}^T\mathbf{PL}$ by inspection is then

$$(\mathbf{L}^T\mathbf{PL})^+ \;=\; \begin{bmatrix} \mathbf{V}_p & \mathbf{V}_o \end{bmatrix} \begin{bmatrix} \Lambda_p^{-2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_p^T \\ \mathbf{V}_o^T \end{bmatrix} \tag{8}$$

Therefore

$$(\mathbf{PL})^+ \;=\; (\mathbf{L}^T\mathbf{PL})^+(\mathbf{L}^T\mathbf{P}) \tag{9}$$

gives the relationship between the two pseudoinverses $(\mathbf{PL})^+$ and $(\mathbf{L}^T\mathbf{PL})^+$.

The process of applying the pseudoinverse operator can be broken down into the following steps:

1) Apply the transpose $\mathbf{L}^T\mathbf{P}$,
2) Transform into the eigenvalue space of $\mathbf{L}^T\mathbf{PL}$ using $\mathbf{V}^T$,
3) Invert the nonzero eigenvalues of Λ_p,
4) Inverse transform via $\mathbf{V}$.

If the singular value decomposition of an operator is known, this procedure gives the pseudoinverse. The solution will have no nonzero component in the null space of the operator **PL**. The dimensions of linear operators are proportional to the size of the data set that is being operated on. Considering the size of the dimensions in the case of seismic processing, it would be impossible to generally determine the singular value decomposition of any operator. But there are exceptions: the operator may separate into smaller dimensional pieces, or the eigenvector space may happen to coincide with Fourier transform space. Such is the case for slant stacking, but only for particular forms of **P**. The next section will deal with slant stacks: what they are, and the effects that applying a projector **P** have on the data before stacking.

The alternative to the pseudoinverse, that of adding a small value to the diagonal of the operator $\mathbf{L}^T\mathbf{PL}$, easily removes the problem of the null space. For example, consider what happens to the SVD when a constant $\alpha^2\mathbf{I}$ is added to $\mathbf{L}^T\mathbf{PL}$:

$$\mathbf{L}^T\mathbf{PL} + \alpha^2\mathbf{I} = \begin{bmatrix}\mathbf{V}_p & \mathbf{V}_o\end{bmatrix}\begin{bmatrix}\Lambda_p^2+\alpha^2\mathbf{I} & 0 \\ 0 & \alpha^2\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{V}_p^T \\ \mathbf{V}_o^T\end{bmatrix}. \qquad (10)$$

Since all the singular values are now positive, the least squares operator is nonsingular. If α^2 is small, the inverse is approximately

$$(\mathbf{L}^T\mathbf{PL} + \alpha^2\mathbf{I})^{-1} \approx \begin{bmatrix}\mathbf{V}_p & \mathbf{V}_o\end{bmatrix}\begin{bmatrix}\Lambda_p^{-2} & 0 \\ 0 & \alpha^{-2}\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{V}_p^T \\ \mathbf{V}_o^T\end{bmatrix} \qquad (11)$$

This operator will behave properly only if the data has no energy residing in the null space, otherwise α^{-2} will cause a large amplification of these components. If the data actually resulted from the application of **PL** [equation (4)] to some model, it would have exactly nothing in the null space, and it would be possible to use (11) in place of the pseudoinverse (7) in order to arrive at the same solution. If (11) is accurate, the choice of whether to use (7) or (11) turns out to be a choice of convenience. Note that adding non-constant positive values to the diagonal of $\mathbf{L}^T\mathbf{PL}$ instead of a constant α^2 will change not only the singular values Λ but the unitary matrix $\mathbf{V}$ as well.

It was stated above that if a vector **d** has no component in the null space of $\mathbf{L}^T\mathbf{PL}$, it makes no difference whether it were fed into the operator (11) or the pseudoinverse (7); the same output would be obtained. It would be hoped that a similar situation holds between the pseudoinverse $(\mathbf{L}^T\mathbf{PL})^+$ and the inverse of the full operator $\mathbf{L}^T\mathbf{L}$, but in general it does not. Assuming that **L** has an inverse (in comparison to **PL** which does not), $(\mathbf{L}^T\mathbf{L})^{-1}$ exists. The question is whether $(\mathbf{L}^T\mathbf{L})^{-1}$ may be used in place of the pseudoinverse $(\mathbf{L}^T\mathbf{PL})^+$ to obtain the same output **u**. This is true only if the two operators share the same eigenvectors, that is, when they both have the same unitary **V** matrix in their singular value decompositions. Upon first thought this may seem to be a hopelessly stringent requirement, but we shall see that for slant stacks this requirement can be satisfied, but in order for this to be so the truncation projector **P** must have a particular structure.

Under these favorable conditions one has a choice of inverse operators to apply. Here we assume **d** arises from an application of the forward operator:

$$\mathbf{d} = \mathbf{PLu} + \mathbf{n} \tag{1}$$

where **n** is an independent noise term. Ideally **n** should be zero so that no null space components are introduced into **d**. But if noise is allowed to add into **d**, $\mathbf{L}^T\mathbf{P}$ may be applied to annihilate any component of **d** in the null space. The following theorem summarizes these facts: (Theorem A)

Let $\mathbf{d} = \mathbf{PLu}$ *for some* **u** *(i.e. noise* $\mathbf{n} = 0$ *in equation* (1) *). Then* **d** *has no nonzero component in the null space of* **PL***, which is the space represented by the zeros of* **P**. *Also, for positive scalar* α*, the following two systems yield the same estimate* **u** *as* $\alpha \to 0$*:*

$$\mathbf{u} = (\mathbf{L}^T\mathbf{PL} + \alpha^2\mathbf{I})^{-1}\mathbf{L}^T\mathbf{d}, \tag{12}$$

$$\mathbf{u} = (\mathbf{L}^T\mathbf{PL})^+\mathbf{L}^T\mathbf{d}. \tag{13}$$

Furthermore if $\mathbf{L}^T\mathbf{L}$ *shares eigenvectors with* $\mathbf{L}^T\mathbf{PL}$*, then the estimate* **u** *found by*

$$\mathbf{u} = (\mathbf{L}^T\mathbf{L})^{-1}\mathbf{L}^T\mathbf{d} \tag{14}$$

is equal to the estimates **u** *given in equations (12) and (13).*

For the proof of the last statement, recognize that $(\mathbf{L}^T\mathbf{L})^{-1}$ can be expanded as

$$(\mathbf{L}^T\mathbf{L})^{-1} = \begin{bmatrix}\mathbf{V}_p & \mathbf{V}_o\end{bmatrix}\begin{bmatrix}\Lambda_p^{-2} & 0\\ 0 & \Lambda_o^{-2}\end{bmatrix}\begin{bmatrix}\mathbf{V}_p^T\\ \mathbf{V}_o^T\end{bmatrix} \tag{15}$$

Comparing this to the eigenvalue expansion of $(\mathbf{L}^T\mathbf{PL})^+$ given in equation (8), the effect of the projection **P** is simply to replace the eigenvalues Λ_o with zero values. Since **d** has no noise, $\mathbf{L}^T\mathbf{d} = \mathbf{L}^T\mathbf{Pd}$. Inserting (15) and (5), the SVD of $\mathbf{L}^T\mathbf{P}$, into the expression (14) for **u** yields

$$\mathbf{u} = \begin{bmatrix} \mathbf{V}_p & \mathbf{V}_o \end{bmatrix} \begin{bmatrix} \Lambda_p^{-2} & 0 \\ 0 & \Lambda_o^{-2} \end{bmatrix} \begin{bmatrix} \Lambda_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_p^T \\ \mathbf{U}_o^T \end{bmatrix} \mathbf{d} \quad (16)$$

Multiplying the two diagonal matrices together will zero out the Λ_o eigenvalues, making equation (16) equivalent to (13). This proves the theorem.

Slant stacking and dip filtering

This section will illustrate the ideas introduced above with the choice of slant stacking for the operator L. Suppose we have a two dimensional data set, denoted by **u**, which is indexed by the two parameters p and τ. Let p be the horizontal axis, having the dimensions of slowness (inverse velocity). Let τ have the dimension of time. The slant stack operator **L** is defined to be the linear operation

L:
$$w(h,t) = \int_{-\infty}^{\infty} dp \; u(p, \tau = t - ph) \quad (17)$$

The index h is a spatial index: it has the dimension of length, and in this paper it will assume the role of offset from the midpoint of a gather. Typically the sampling in time is sufficiently dense that the nyquist frequency happens to be greater than the highest frequency on the data, assuming that the data has gone through a filtering stage. On the other hand the sampling in p or h, for actual seismic data, is coarse and narrowly limited. The slant stack definition (17) assumes that $u(p,\tau)$ is continuously sampled in p. The input $u(p,\tau)$ may be arbitrary, but assume that the range over which it is nonzero is bounded:

$$u(p,\tau) = 0 \qquad \text{for } |p| > p_{max} \quad (18)$$

While the input **u** is to be bounded in the range of p, the output **w** is not: there can always be found a nonzero $w(h,t)$ for an arbitrarily large h. This is where the projection **P** comes in. It restricts the output of (17) to lie within a certain h range:

P:
$$P(h,t) \equiv \begin{cases} 1 & h_1 \le h \le h_2 \\ 0 & otherwise \end{cases} \quad (19)$$

PL:
$$w(h,t) = \begin{cases} \int_{-\infty}^{\infty} dp \; u(p, \tau = t - ph) & h_1 \le h \le h_2 \\ 0 & otherwise \end{cases} \quad (20)$$

The adjoint of the operator **PL** is, with respect to a normally defined inner product space, is

$\mathbf{L}^T\mathbf{P}$:
$$u(p,\tau) = \int_{h_1}^{h_2} dh \; w(h, t = \tau + ph) \quad (21)$$

A "normal" inner product space has an inner product of the form

$$(\mathbf{u}_1,\mathbf{u}_2) \equiv \iint_{-\infty}^{\infty} dp \, d\tau \, u_1(p,\tau) \, u_2(p,\tau). \tag{22}$$

The normal operation of slant stacking is equivalent to the adjoint $\mathbf{L}^T\mathbf{P}$ defined above. It is viewed as stacking each constant-h trace of a data set $w(h,t)$ at different slopes p, the data being bounded by the offsets h_1 and h_2. If a seismic data set happens to consist of a number of constant dipping coherent events, it may be accurately modeled by a slant stack (17) of a sparse function $u(p,\tau)$. Therefore it is more appropriate, in slant stacking seismic data, to attempt to recover the inverse $\mathbf{u} = \mathbf{L}^+\mathbf{w}$ rather than do the forward operation $\mathbf{u} = \mathbf{L}^T\mathbf{w}$. The latter operation will contain all of the truncation artifacts introduced by the finite aperture $h_1 \le h \le h_2$. The ideal solution $\mathbf{u}$ should be independent of the aperture in h, but this depends on how valid the model $\mathbf{w} = \mathbf{Lu}$ really is. Therefore, we may summarize that *one objective in slant stacking is to reduce truncation effects due to the finite aperture in h of the data. This may be done by applying the pseudoinverse operator* $\mathbf{L}^+$ *to the data in place of the standard slant stack operator* $\mathbf{L}^T$.

From the previous section, the pseudoinverse is $\mathbf{L}^+ = (\mathbf{L}^T\mathbf{PL})^+\mathbf{L}^T$, and so the pseudoinverse may be implemented by first stacking the data with $\mathbf{L}^T$ (equation (17)), followed by the filter $(\mathbf{L}^T\mathbf{PL})^+$. It will be shown that this latter operator reduces to the "rho filter" $|\omega|$ as the aperture of the data expands to infinity: $h_1 \to -\infty$, $h_2 \to \infty$. In theory the rho filter yields an exact inverse $\mathbf{L}^{-1}$ to the slant stack operator, but of course in practice the data will always be severely limited in the horizontal dimension h.

Estimating the slant stack pseudoinverse

This section will be concerned with deriving $\mathbf{L}^T\mathbf{PL}$ and $(\mathbf{L}^T\mathbf{PL})^+$ for slant stacks. The expressions defining $\mathbf{L}$, $\mathbf{L}^T$, and $\mathbf{P}$ are those of the last section: equations (17), (19) and (20). The pseudoinverse can be found because of the simple structure (19) imposed on the truncation $\mathbf{P}$; the orthogonal transformations in the singular value decomposition of $\mathbf{L}^T\mathbf{PL}$ turn out to be Fourier transforms.

The response $\mathbf{Lu}$ to an impulse $u(p,\tau) = \delta(p-\tilde{p})\delta(\tau-\tilde{\tau})$ is

$$w(h,t) = \int_{-\infty}^{\infty} dp \;\; \delta(p-\tilde{p})\delta(t-ph-\tilde{\tau})$$

$$= \delta(t-\tilde{p}h-\tilde{\tau}) \tag{23}$$

Applying $\mathbf{L}^T\mathbf{P}$ to this gives the impulse response of $\mathbf{L}^T\mathbf{PL}$:

$$u(p,\tau) = \int_{h_1}^{h_2} dh\ \delta(\tau + ph - \tilde{p}h - \tilde{\tau})$$

$$= \int_{h_1}^{h_2} dh\ \frac{\delta(h-h_0)}{|p-\tilde{p}|} \qquad (24)$$

where $h_0 \equiv -(\tau - \tilde{\tau})/(p-\tilde{p})$. Thus

$$u(p,\tau) = \begin{cases} |p-\tilde{p}|^{-1} & \text{for } h_1 \le h_0 \le h_2 \\ 0 & otherwise \end{cases}$$

$$= |p-\tilde{p}|^{-1} H(h_0 - h_1)\ H(h_2 - h_0) \qquad (25)$$

where $H(x)$ is the Heaviside step or unit step function. With this result the transformation $\mathbf{L}^T\mathbf{PL}$ may be represented in the form of a double integral with kernel K:

$$u(p,\tau) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\tau\ K(p,\tau;\ \tilde{p},\tilde{\tau})\ \tilde{u}(\tilde{p},\tilde{\tau}) \qquad (26a)$$

in which the kernel is

$$K(p,\tau;\ \tilde{p},\tilde{\tau}) = |p-\tilde{p}|^{-1}\ H\left(-\frac{\tau-\tilde{\tau}}{p-\tilde{p}} - h_1\right) H\left(h_2 + \frac{\tau-\tilde{\tau}}{p-\tilde{p}}\right) \qquad (26b)$$

Note that K is convolutional in both τ and p, therefore the filter is multiplicative in two dimensional Fourier transform (2DFT) space. In other words, the operator $\mathbf{L}^T\mathbf{PL}$ is diagonalized by a 2DFT. With respect to the Fourier transform, it is important that the forward and inverse transforms be adjoints of each other; the singular value decomposition requires that $\mathbf{V}$ and $\mathbf{V}^T$ be adjoint. The transform convention used here is

$$\tilde{u}(\xi,\eta) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\tau\ u(p,\tau)\ e^{-i\xi p - i\eta\tau}, \qquad (27a)$$

$$u(p,\tau) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\ \tilde{u}(\xi,\eta)\ e^{+i\xi p + i\eta\tau}. \qquad (27b)$$

In what follows, the forward 2DFT is identified with the operator $\mathbf{V}^T$ while the inverse 2DFT is identified with **V**.

To find the filter (26b) in the Fourier domain, Fourier transform the kernel $K(p,\tau)$:

$$\tilde{K}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp\; d\tau\; |p|^{-1} H\left(-\frac{\tau}{p} - h_1\right) H\left(h_2 + \frac{\tau}{p}\right) e^{-i\xi p - i\eta\tau}$$

$$= \frac{1}{2\pi}\left\{ \int_0^{\infty} dp \int_{-h_2 p}^{-h_1 p} d\tau + \int_{-\infty}^{0} dp \int_{-h_1 p}^{-h_2 p} d\tau \right\}\left\{ |p|^{-1} e^{-i\xi p - i\eta\tau} \right\} \tag{28}$$

This partition of the integral keeps the sense of area under the double integral positive. Now let $\tau = hp$, $d\tau = p\; dh$. By change of variable the integral becomes

$$\tilde{K}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \int_{-h_2}^{-h_1} dh\; |p|\; |p|^{-1} e^{-i\xi p - i\eta p h}$$

$$= \int_{-h_2}^{-h_1} dh\; \delta(\xi + \eta h) = \int_{-h_2}^{-h_1} dh\; \frac{\delta(\xi/\eta + h)}{|\eta|} \tag{29}$$

Therefore

$$\tilde{K}(\xi,\eta) = |\eta|^{-1} H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right) \tag{30}$$

where $H(x)$ again is the unit step function, and is used to define the region where the filter is nonzero, that is where the delta function of equation (29) lies within the finite bounds of the integral. The nonzero region of the filter $\tilde{K}(\xi,\eta)$ in the Fourier plane is shown in figure 1. As $h_1 \to -\infty$ and $h_2 \to \infty$, the filter covers the entire Fourier plane.

The pseudoinverse is characterized by taking the inverse of the nonzero portion of the filter in the Fourier domain. The null space of the operator $\mathbf{L}^T\mathbf{P}\mathbf{L}$ is obvious in this case; it comprises events whose dip spectrum lies outside the nonzero range of the filter in figure 1. The pseudoinverse of $\tilde{K}$ is thus

$$\tilde{K}^{+}(\xi,\eta) = |\eta|\, H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right) \tag{31}$$

As the aperture (h_1, h_2) is widened, the filter converges to $|\eta|$, the rho filter. A time-space domain implementation of this filter is preferable, for remember that in practice the spatial axis is discrete and limited. A Fourier transform implementation of the filter in that direction will result in serious wraparound problems. Therefore find an expression for $\tilde{K}^{+}$ in the (p,τ)

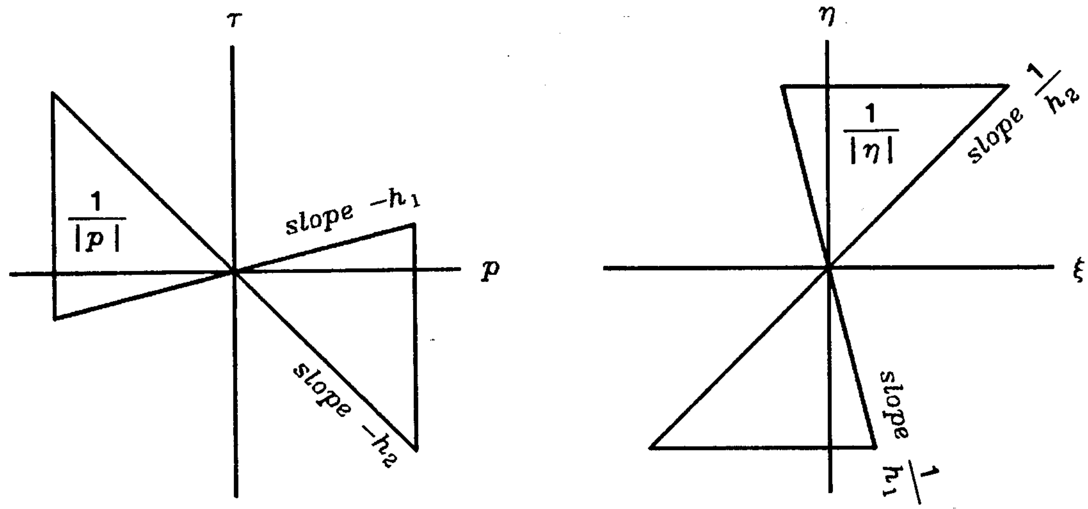

FIG. 1. Slant stack impulse response $\mathbf{L}^T\mathbf{PL}$ in the space domain (equation 26b) and filter in the Fourier domain (equation 30).

domain:

$$K^+(p,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\eta\, d\xi\; |\eta|\, H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right) e^{ip\xi + i\tau\eta} \tag{32}$$

Again let $h = \xi/\eta$ be the new variable of integration so that $\xi = \eta h$ and $d\xi = |\eta|\,dh$. The limits of the integral shall be taken to enclose a positive area.

$$K^+(p,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \int_{h_1}^{h_2} dh\; \eta^2\, e^{i\eta(\tau + hp)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta\; \frac{-i\eta}{p}\, e^{i\eta\tau}\left(e^{i\eta p h_2} - e^{i\eta p h_1}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i\eta}{p}\, e^{i\eta(\tau + ph_2)} d\eta \;-\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i\eta}{p}\, e^{i\eta(\tau + ph_1)} d\eta \tag{33}$$

or,

$$K^+(p,\tau) = \frac{1}{p}\delta'(\tau + ph_2) - \frac{1}{p}\delta'(\tau + ph_1) \tag{34}$$

The filter is depicted in figure 2. It consists of a delta derivative positioned along the slopes $-h_2$ and $-h_1$, with a weight of $1/|p|$ applied. This filter may be implemented by finite differencing at the positions where the delta derivative is. In a way the inverse filter uses only the truncation effects associated with the forward filter K to do its work. With the differencing operators positioned along the edges of the filter as they are, the inverse kernel K^+ in (34) will detect variations of the input field at slopes $-h_1$ and $-h_2$.

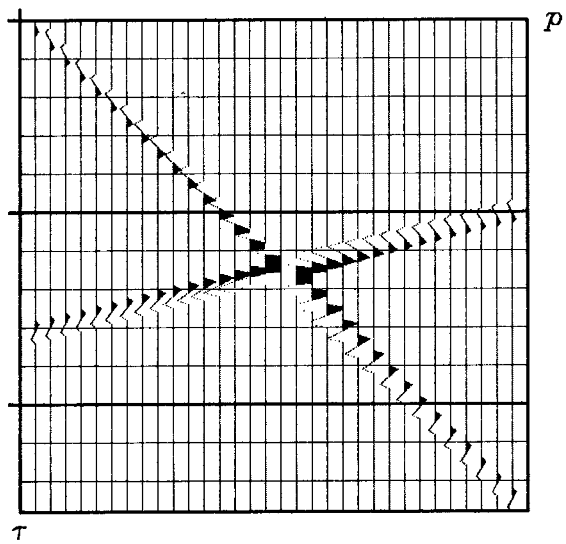

FIG. 2. Slant stack impulse response $(\mathbf{L}^T\mathbf{P}\mathbf{L})^+$ in the space domain (equation 34).

For the infinite aperture case, $h_1 \to -\infty$ and $h_2 \to \infty$, the filter reduces to the familiar one-dimensional "rho filter". This is seen by inverse Fourier transforming the expression (31) with the unit step functions absent from the integrand:

$$K^+(p,\tau) = \delta(p)\,\rho(\tau) \qquad (35)$$

where $\rho(\tau)$ is the inverse transform of $|\eta|$, the variable η being a temporal frequency. Strictly speaking the inverse transform of $|\eta|$ doesn't exist, but discrete implementations of it do and can easily be found. Two ways of implementing the rho filter are possible.

The obvious way to design a discrete rho filter is to inverse transform into the discrete domain. That is,

$$\rho(\tau) = \int_{-\pi/\Delta\tau}^{\pi/\Delta\tau} d\eta\ |\eta|\, e^{i\eta\tau} \qquad (36)$$

The limits of integration are bounded by the temporal Nyquist frequency $\pi/\Delta\tau$. Performing this transform yields

$$\rho(\tau) = \begin{cases} -\dfrac{4}{\Delta\tau^2 j_\tau^{\,2}} & j_\tau \ even \\ 0 & j_\tau \ odd \\ \dfrac{2\pi^2}{\Delta t^2} & j_\tau \ zero \quad (\tau = j_\tau \Delta\tau) \end{cases} \tag{37}$$

The alternative is to calculate the continuous transform at τ values where it exists. In this case, the integral exists everywhere except at $\tau = 0$.

$$\rho(\tau) = \int_{-\infty}^{\infty} d\eta \, |\eta| e^{i\eta\tau} = \frac{-2}{\tau^2} \qquad \tau \neq 0 \tag{38}$$

Since the filter $|\eta|$ can have no DC frequency component, the remaining unknown filter element $\tau = 0$ may be chosen as the negative average of the other components of the filter in order to satisfy this constraint:

$$\rho_0 = \sum_{j \neq 0} \Delta\tau \, \frac{2}{j^2 \Delta\tau^2} \tag{39}$$

To summarize this section, recognizing the null space of $\mathbf{L}^T\mathbf{P}\mathbf{L}$ allows one to see what events are unrecoverable from slant stack inversion. Specifically, any events having slopes outside the range of the slopes stacked at are eliminated in the forward stack and cannot be recovered by the inverse. Apart from these events the original data may be reconstructed by the following process:

(a) Given $\mathbf{d}$, apply the transpose slant stack $\mathbf{L}^T\mathbf{P}\mathbf{d}$.

(b) Apply a filter that is of the form of the pseudoinverse $(\mathbf{L}^T\mathbf{P}\mathbf{L})^+$. Now the slant stack operator satisfies the conditions of theorem A of the previous section, so it suffices to apply the whole-space inverse $\rho(\tau)$ of equations (38), (39).

Table I summarizes the forward and inverse filters developed in this section for slant stacks. For completeness, the operator $\mathbf{L}\mathbf{P}'\mathbf{L}^T$ and its pseudoinverse is included in the table, which can be used to invert the slant stack from (h,t) space to (p,τ) space. The projector $\mathbf{P}'$ limits the slopes p to lie between p_1 and p_2.

Table I: Slant Stack Filters

Filter	Dip Range	Space Domain	Fourier Domain[1]
$\mathbf{L}^T\mathbf{PL}$	h_1, h_2	$\lvert p\rvert^{-1} H\left(-\frac{\tau}{p} - h_1\right) H\left(h_2 + \frac{\tau}{p}\right)$	$\lvert\eta\rvert^{-1} H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right)$
	∞	$\lvert p\rvert^{-1}$	$\lvert\eta\rvert^{-1}$
$(\mathbf{L}^T\mathbf{PL})^+$	h_1, h_2	$\frac{1}{p}\delta'(\tau + ph_2) - \frac{1}{p}\delta'(\tau + ph_1)$	$\lvert\eta\rvert H\left(\frac{\xi}{\eta} - h_1\right) H\left(h_2 - \frac{\xi}{\eta}\right)$
	∞	$\frac{-2}{\tau^2}\delta(p)$ (note 2)	$\lvert\eta\rvert$
$\mathbf{LP'L}^T$	p_1, p_2	$\lvert h\rvert^{-1} H\left(-\frac{t}{h} - p_1\right) H\left(p_2 + \frac{t}{h}\right)$	$\lvert\omega\rvert^{-1} H\left(-\frac{k}{\omega} - p_1\right) H\left(p_2 + \frac{k}{\omega}\right)$
	∞	$\lvert h\rvert^{-1}$	$\lvert\omega\rvert^{-1}$
$(\mathbf{LP'L}^T)^+$	p_1, p_2	$\frac{1}{h}\delta'(t - hp_2) - \frac{1}{h}\delta'(t - hp_1)$	$\lvert\omega\rvert H\left(-\frac{k}{\omega} - p_1\right) H\left(p_2 + \frac{k}{\omega}\right)$
	∞	$\frac{-2}{t^2}\delta(h)$ (note 2)	$\lvert\omega\rvert$

Notes:

1. Fourier variables: $\tau \to \eta$, $p \to \xi$; $t \to \omega$, $h \to k$

2. The filter point at zero t or τ is taken to cancel the mean of the filter.

Velocity stacks and slowness stacks

The important features of the pseudoinverse theory introduced in the first section of this paper have now been illustrated with the slant stack operator. A more appropriate linear transformation to use on seismic data, specifically common midpoint gathers (CMGs), is the operation of normal moveout and stack. For example, one method of velocity analysis applied to a CMG that uses no elaborate semblance measure is to sum along hyperbolas corresponding to the selected range of velocities, each trace being weighted uniformly in the sum. The similarity to slant stacking is obvious; let us call this operation velocity stacking. Let the definition of the velocity stack operator L be

$$w(h,t) \;=\; \int_{v_1}^{v_2} dv \; u(v, \sqrt{t^2 - h^2/v^2}) \tag{40}$$

The function $w(h,t)$ represents the wavefield in offset space, indexed by offset h and time t, while the function $u(v,\tau)$ represents a velocity space or velocity panel. It is understood that the integrand is defined zero when the square root turns imaginary:

$$u(v, \sqrt{t^2 - h^2/v^2}) \;\equiv\; 0 \qquad t < h/v \tag{41}$$

The limits of the integral may be replaced by $\pm\infty$ if it is further understood that the function $u(v,\tau)$ is zero outside a certain range of time and velocity. As before, the projector **P** will be defined as truncation of the wavefield over the finite offset aperture $h_1 \le h \le h_2$. This represents the range of offsets over which the gather was recorded. The adjoint $\mathbf{L}^T\mathbf{P}$ of equation (40) is

$$u(v,\tau) \;=\; \int_{h_1}^{h_2} dh \; \frac{\tau}{\sqrt{\tau^2 + h^2/v^2}} \; w(h, \sqrt{\tau^2 + h^2/v^2}) \tag{42}$$

This was derived from the definition of an adjoint operator

$$(\mathbf{u}, \mathbf{L}^T\mathbf{P}\mathbf{w})_V \;=\; (\mathbf{w}, \mathbf{P}\mathbf{L}\mathbf{u})_H \tag{43}$$

in which $(\tilde{\mathbf{u}}, \mathbf{u})_V$ is an inner product in velocity space of the form of equation (22). Likewise it is assumed that a corresponding inner product $(\tilde{\mathbf{w}}, \mathbf{w})_H$ has been defined in offset space.

Rather than continue with the development of the pseudoinverse with the velocity stack equations (40) and (42), let us turn to slowness stacks. The two definitions differ only in the spaces chosen to uniformly sample from. In contrast to the velocity stack, the slowness stack is defined to sample even increments of slowness $p \equiv 1/v$. Let the slowness stack **L** be the operation

$$w(h,t) \;=\; \int_0^{\infty} d\tau \int_0^{\infty} dp \; \delta(\tau - \sqrt{t^2 - p^2h^2}) \; u(p,\tau) \tag{44}$$

The determination of the adjoint $\mathbf{L}^T$ depends on the definition of inner product in slowness space and offset space. Let the inner product in slowness space (p,τ) be weighted by τ:

$$(\tilde{\mathbf{u}}, \mathbf{u})_P \;\equiv\; \int_0^{\infty} dp \int_0^{\infty} \tau \, d\tau \; \tilde{u}(p,\tau) \; u(p,\tau) \tag{45}$$

A similar inner product definition (weighting by t) applies to offset space. As a result the adjoint $\mathbf{L}^T$ is found to be symmetric in form to (45):

$$u(p,\tau) = \int_0^\infty dt \int_0^\infty dh\, \delta(t - \sqrt{\tau^2 + p^2h^2})\, w(h,t) \tag{46}$$

The offset and slowness spaces are defined to be the quarter planes $0 \le h < \infty$, $0 \le t < \infty$ and $0 \le p < \infty$, $0 \le \tau < \infty$. Only positive values of slowness have any physical meaning, and since the moveout formulas are symmetric for positive and negative offsets h it is also sufficient to restrict h to being positive.

The pseudoinverse of the slowness stack

The first step, as before, is to put the kernel of $\mathbf{L}^T\mathbf{PL}$ in convolutional form so that the kernel of the pseudoinverse may be determined in Fourier space. From equation (45), the response $\mathbf{L}\delta(p-\tilde{p})\delta(\tau-\tilde{\tau})$ is

$$w(h,t) = \delta(\sqrt{t^2 - \tilde{p}^2h^2} - \tilde{\tau})$$

$$= \frac{\tilde{\tau}}{t}\delta(t - \sqrt{\tilde{\tau}^2 + \tilde{p}^2h^2}) \tag{47}$$

The impulse response $\mathbf{L}^T\mathbf{PL}\delta(p-\tilde{p})\delta(\tau-\tilde{\tau})$ is, by (46),

$$u(p,\tau) = \int_{h_1}^{h_2} dh\, \frac{\tilde{\tau}}{\tilde{t}}\delta(t - \tilde{t}) \qquad \begin{cases} t \equiv \sqrt{\tau^2 + p^2h^2} \\ \tilde{t} \equiv \sqrt{\tilde{\tau}^2 + \tilde{p}^2h^2} \end{cases} \tag{48}$$

The delta function $\delta(t - \tilde{t})$ is nonzero for a unique offset $h^2 = h_0^2$. Transform it into a delta function in h in order to invoke the sifting property of deltas:

$$u(p,\tau) = \int_{h_1}^{h_2} dh\, \frac{\tilde{\tau}}{\tilde{t}}\frac{t}{h_0}|p^2 - \tilde{p}^2|^{-1}\delta(h - h_0) \qquad \left[h_0 \equiv -\frac{\tau^2 - \tilde{\tau}^2}{p^2 - \tilde{p}^2}\right] \tag{49}$$

Now at $h = h_0$, $\tilde{t}$ is equal to t. The integrand sifts out, with the exception that the integral is zero if h_0 falls outside the range of integration. Taking this into account, and substituting for h_0 , $\mathbf{L}^T\mathbf{PL}\delta$ becomes

$$u(p,\tau) = \tilde{\tau}\,|p^2 - \tilde{p}^2|^{-1/2}|\tau^2 - \tilde{\tau}^2|^{-1/2} H\!\left(h_0^2 - h_1^2\right) H\!\left(h_2^2 - h_0^2\right) \tag{50}$$

where $H(x)$ is the unit step function. The nonzero parts of the filter are bounded by the curves $h_0 = h_1$ and $h_0 = h_2$ which pass through the origin, h_0 being a function of p and τ.

With the impulse response (50), the transformation $\mathbf{L}^T\mathbf{PL}$ may be put in integral form:

$$u(p,\tau) = \iint_0^\infty d\tilde{p}\, d\tilde{\tau}\, \tilde{\tau}\, K(p^2 - \tilde{p}^2, \tau^2 - \tilde{\tau}^2)\, \tilde{u}(\tilde{p},\tilde{\tau}) \tag{51a}$$

where

$$K(x,y) = |xy|^{-1/2} H\left(-\frac{y}{x} - h_1^2\right) H\left(h_2^2 - \frac{y}{x}\right) \tag{51b}$$

The kernel K is illustrated in figure 3. The variables of integration now may be changed in an obvious way to form a convolutional integral.

$$u(x^{1/2}, y^{1/2}) = \iint_0^\infty \frac{d\tilde{x}}{2\tilde{p}} \frac{d\tilde{y}}{2\tilde{\tau}} \tilde{\tau}\, K(x - \tilde{x}, y - \tilde{y})\, \tilde{u}(\tilde{x}^{1/2}, \tilde{y}^{1/2}) \tag{51c}$$

The pseudoinverse of this rescaled operator may be found just as in the case of slant stacking: transform the kernel to the Fourier domain, invert the singular values of the nonzero part, and inverse transform. The change of variables implemented above is an invertible linear transformation **D**, allowing the operator to be diagonalized by the (unitary) Fourier transform:

$$\mathbf{L}^T\mathbf{PLu} = \mathbf{D}^{-1}\mathbf{U}\Lambda\mathbf{U}^T\mathbf{DSu} \tag{52}$$

The change of variables $x = p^2$ and $y = \tau^2$ is identified by **D**. $\mathbf{U}^T$ is the Fourier transform. Λ is diagonal; it holds the values of the filter in Fourier space. There is an additional diagonal **S** which represents the prescaling of $\tilde{u}$ by $1/4\tilde{p}$ in equation (51c). Strictly speaking this decomposition of $\mathbf{L}^T\mathbf{PL}$ is not a singular value decomposition, but if $\mathbf{L}^T\mathbf{PL}$ happens to have a true inverse, it will be uniquely given by

$$(\mathbf{L}^T\mathbf{PL})^{-1} = \mathbf{S}^{-1}\mathbf{D}^{-1}\mathbf{U}\Lambda^{-1}\mathbf{U}^T\mathbf{D} \tag{53}$$

An approximation to the pseudoinverse then is gained by replacing Λ^{-1} by $\mathbf{P}_1\Lambda$ whose values in the null subspace are annihilated by $\mathbf{P}_1$.

$$(\mathbf{L}^T\mathbf{PL})^{+} \equiv \mathbf{S}^{-1}\mathbf{D}^{-1}\mathbf{U}\mathbf{P}_1\Lambda^{-1}\mathbf{U}^T\mathbf{D} \tag{54}$$

If the system (54) is not a singular value decomposition, then theorem A of the first section does not apply, and we cannot immediately substitute $(\mathbf{L}^T\mathbf{L})^+$ in for $(\mathbf{L}^T\mathbf{PL})^+$ in the generalized inverse solution of $\mathbf{u} = (\mathbf{L}^T\mathbf{PL})^+\mathbf{L}^T\mathbf{Pd}$ and expect to get the same estimate for **u**. The reason for wanting to do so is strictly one of efficiency; $(\mathbf{L}^T\mathbf{L})^+$ may be easier to implement than $(\mathbf{L}^T\mathbf{PL})^+$. However the substitution may be made if the data **d** satisfies the relation $\mathbf{d} = \mathbf{PL}\tilde{\mathbf{u}}$ well, i.e. when there is little or no noise. The proof follows.

It is proposed that the estimate **u** be made in either of two ways: $(\mathbf{L}^T\mathbf{L})^+\mathbf{L}^T\mathbf{Pd}$ or $(\mathbf{L}^T\mathbf{PL})^+\mathbf{L}^T\mathbf{Pd}$. Both are special cases of the estimate

$$\mathbf{u} \;=\; \left[\mathbf{S}^{-1}\mathbf{D}^{-1}\mathbf{U}\mathbf{P}_2\Lambda^{+}\mathbf{U}^{T}\mathbf{D}\right]\left[\mathbf{D}^{-1}\mathbf{U}\mathbf{P}_1\Lambda\mathbf{U}^{T}\mathbf{D}\mathbf{S}\right]\tilde{\mathbf{u}}$$

$$= \mathbf{S}^{-1}\mathbf{D}^{-1}\mathbf{U}\mathbf{P}_2\Lambda^{+}\mathbf{P}_1\Lambda\mathbf{U}^{T}\mathbf{D}\mathbf{S}\tilde{\mathbf{u}} \tag{55}$$

where $\mathbf{P}_1$, $\mathbf{P}_2$ are diagonal projection matrices representing clipping in the Fourier domain. The data $\mathbf{d} = \mathbf{L}\tilde{\mathbf{u}}$ is assumed to come from $\tilde{\mathbf{u}}$. $\mathbf{P}_1$ corresponds to the original $\mathbf{P}$. Now $\mathbf{L}^T\mathbf{L}$ is the version of $\mathbf{L}^T\mathbf{P}\mathbf{L}$ without any truncation which corresponds to $\mathbf{P}_2 = \mathbf{I}$ in equation (54). Otherwise $\mathbf{P}_2 = \mathbf{P}_1$. As a matter of fact all operators derived from projections $\mathbf{P}$ defined by limiting the aperture in h are diagonalized by equations like (52), and vary only in the scope of the clipping $\mathbf{P}_{1,2}$ made in the Fourier domain. Finally note that $\mathbf{P}_2$ in equation (55) may be replaced by either $\mathbf{P}_1$ or the identity $\mathbf{I}$, and the resulting $\mathbf{u}$ is unaffected. To summarize, *when no noise is present* ($\mathbf{d} = \mathbf{L}\mathbf{u}$) *an identical estimate* $\mathbf{u}$ *is obtained by applying* $(\mathbf{L}^T\mathbf{L})^+$ *in place of* $(\mathbf{L}^T\mathbf{P}\mathbf{L})^+$ *in equation* (54).

The Fourier transform of the kernel K in equation (51b) turns out to have a simple expression:

$$\tilde{K}(\xi,\eta) \;=\; \frac{1}{2\pi}\iint_{-\infty}^{\infty} dx\; dy\; |xy|^{-1/2} H\!\left(-\frac{y}{x} - h_1^2\right) H\!\left(h_2^2 + \frac{y}{x}\right) e^{-i\xi x - i\eta y}$$

$$= |\xi\eta|^{-1/2} H\!\left(\frac{\xi}{\eta} - h_1^2\right) H\!\left(h_2^2 - \frac{\xi}{\eta}\right) \tag{56}$$

The pseudoinverse is defined to be

$$\tilde{K}^{+}(\xi,\eta) \;\equiv\; |\xi\eta|^{1/2} H\!\left(\frac{\xi}{\eta} - h_1^2\right) H\!\left(h_2^2 - \frac{\xi}{\eta}\right) \tag{57}$$

Rather than continue with the limited aperture version of the pseudoinverse, we shall concentrate on $(\mathbf{L}^T\mathbf{L})^+$. To reiterate, the use of the expanded aperture version is justified only if the data satisfies model (1) with $\mathbf{n} = 0$. In this case,

$$\tilde{K}^{+}(\xi,\eta) \;=\; |\xi\eta|^{1/2} H(\xi\eta) \tag{58}$$

and inverse transforming,

$$K^{+}(x,y) \;=\; \frac{1}{4}|xy|^{-3/2} H(-xy) \qquad \begin{matrix} x \neq 0 \\ y \neq 0 \end{matrix} \tag{59}$$

As in the case of slant stacking, the inverse transform of (58) does not exist for all (x,y); specifically at points where $x = 0$, $y = 0$. But a discrete implementation of it will exist, since the infinite limits on the integral will be replaced by finite limits. The easiest way in this case to determine the filter coefficients at points $x = 0$ and $y = 0$ is to use the constraints imposed by the zero-frequency components of the filter. These constraints are

$$\tilde{K}^+(\xi,\eta) = 0 \quad \text{for } \xi=0 \qquad \int_{-\infty}^{\infty} dx\; K^+(x,y) = 0 \quad \text{for all } y$$

$$\tilde{K}^+(\xi,\eta) = 0 \quad \text{for } \eta=0 \qquad \int_{-\infty}^{\infty} dy\; K^+(x,y) = 0 \quad \text{for all } x \tag{60}$$

Since the only coefficients in question are those at $x = 0$ and $y = 0$, they can be solved in terms of the other known coefficients. This should be done at the discrete filter design stage.

Table II: Slowness Stack Filters

Filter	Dip Range	Kernel $K(p,\tau;\tilde{p},\tilde{\tau})$
$\mathbf{L}^T\mathbf{PL}$	h_1, h_2	$\tilde{\tau}\,\lvert p^2-\tilde{p}^2\rvert^{-1/2}\lvert\tau^2-\tilde{\tau}^2\rvert^{-1/2}\,H\left[h_0^2-h_1^2\right]H\left[h_2^2-h_0^2\right]$ (Note 1)
$(\mathbf{L}^T\mathbf{L})^+$	0, ∞	$4p\tilde{p}\tilde{\tau}\,\lvert p^2-\tilde{p}^2\rvert^{-3/2}\lvert\tau^2-\tilde{\tau}^2\rvert^{-3/2}\,H\left(-\frac{\tau^2-\tilde{\tau}^2}{p^2-\tilde{p}^2}\right)$ (Note 2)

Notes:

1. $h_0^2 \equiv -\dfrac{\tau^2-\tilde{\tau}^2}{p^2-\tilde{p}^2}$

2. The values of the discrete kernel at $p = \tilde{p}$, $\tau = \tilde{\tau}$ are determined by equation (60), $x = p^2$, $y = \tau^2$.

It remains to change variables back into the original coordinates p and τ. The filters in (p,τ) space are summarized in table II for (a) $\mathbf{L}^T\mathbf{PL}$ with restricted aperture $0 \le h_1 \le h \le h_2 < \infty$, and (b) $(\mathbf{L}^T\mathbf{PL})^+$ with wide aperture $0 \le h < \infty$. Figure 3 is a display of responses $\mathbf{L}^T\mathbf{PL}$ to a number of impulses in (p,τ) space. Figure 4 is the response to the same impulses, but calculated by a discrete implementation of the operator **L** followed by $\mathbf{L}^T\mathbf{P}$. The differences between figures 3 and 4 can be ascribed to discretization effects that are not present in the closed form expression (48) for $\mathbf{L}^T\mathbf{PL}$. Figure 5 shows various impulse

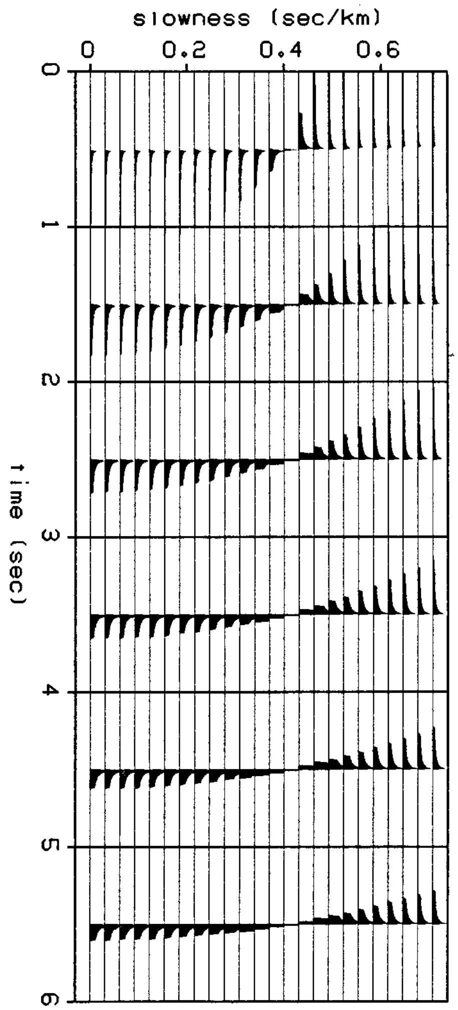

Figure 3

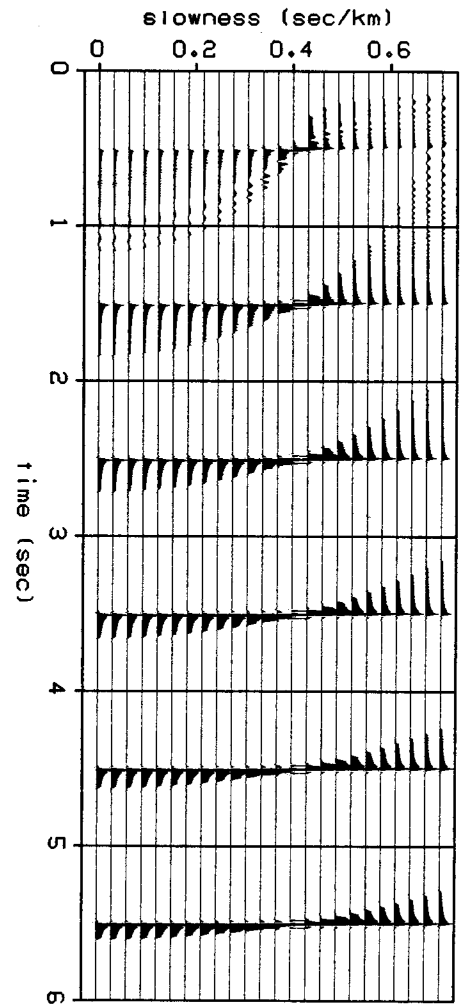

Figure 4

FIG. 3. Responses of $\mathbf{L}^T\mathbf{PL}$ to various impulses. The offset aperture defining **P** is $0.2 \le h \le 2.6$ km. Spreading of energy is mainly horizontal, in the slowness direction, which indicates a loss of resolution in velocity.

FIG. 4. The same filter $\mathbf{L}^T\mathbf{PL}$ as in figure 3, but this one was generated by a discrete implementation of **PL** followed by $\mathbf{L}^T\mathbf{P}$. Discretization effects (bumps) are seen only at shallow times when compared to figure 3.

responses of $(\mathbf{L}^T\mathbf{L})^+$. Finally, the inverse $(\mathbf{L}^T\mathbf{L})^+$ was applied to the data set of figure 4 in an effort to compress the events back to impulses. The result is shown in figure 6. The sidelobes of the impulses have been reduced significantly.

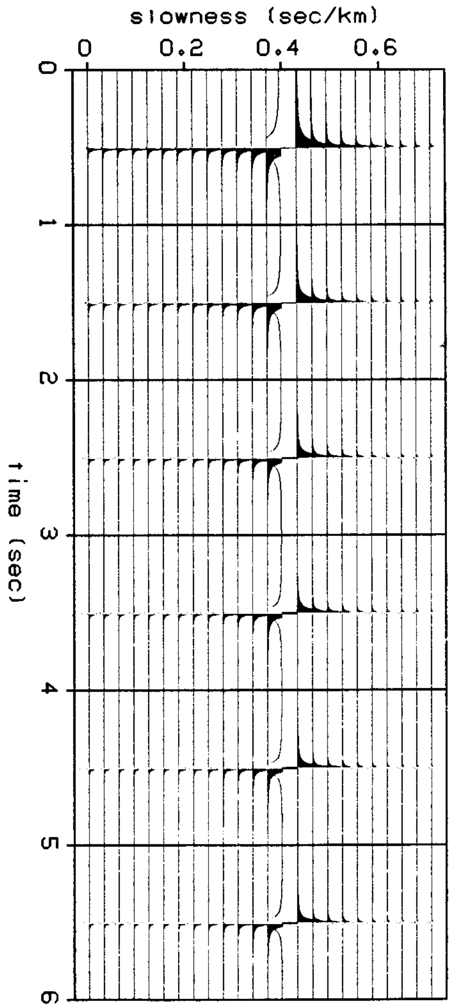

Figure 5

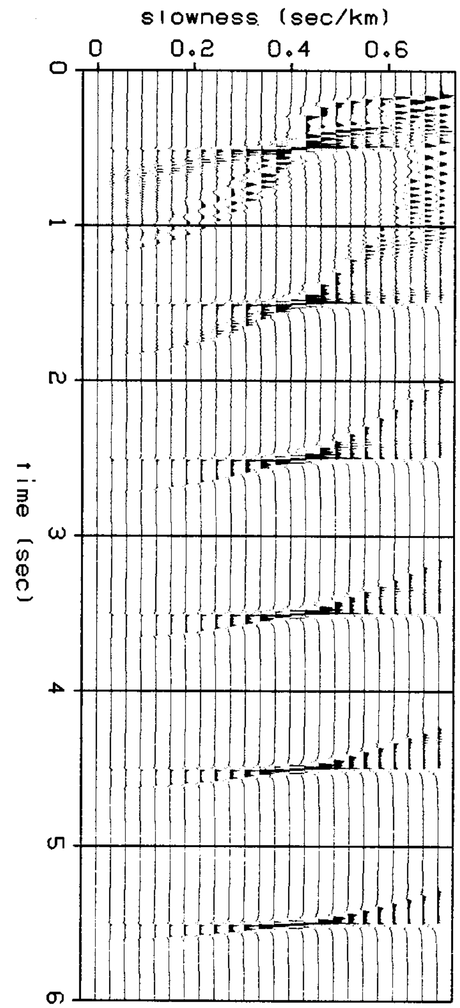

Figure 6

FIG. 5. The response of $(\mathbf{L}^T\mathbf{L})^+$ given in table II, to impulses at $\tilde{p} = 0.4$ and at time points $\tilde{\tau}$ evenly distributed on the time axis. Each response is zero in the upper right and lower left quadrants, positive in the other two quadrants, and has large negative coefficients along the axes $p = \tilde{p}$, $\tau = \tilde{\tau}$, and a large positive coefficient at its origin. The parameters of this filter are consistent with those used to generate figures 3 and 4.

FIG. 6. The application of $(\mathbf{L}^T\mathbf{L})^+$ (figure 5) to figure 4. Artifacts prevail at small times, but much of the energy in the sidelobes of figures 3 and 4 has been concentrated at the spikes. The artifacts could be reduced by a post- low pass filter stage.

Summary: Use of the pseudoinverse

This paper has been concerned with the development of expressions for the pseudoinverses of slant stacking and velocity stacking. These filters may be used in two ways: (a) to obtain a better stack, (b) to obtain an extrapolation via model determination, followed by forward transformation. In the first case, the desired stack may be defined as the inverse operator $\mathbf{L}^+$ applied to the data. It is an attempt at reducing the sidelobes (illustrated in figure 4) which are generated by the normal forward stack operation, $\mathbf{L}^T$. From figure 4 it is easy to see how multiples can severely interfere in the moveout and stack of data at primary velocities.

REFERENCES

Strang, Gilbert, 1980; The pseudoinverse and the singular value decomposition (Section 3.4): Linear Algebra and its Applications, Second Edition, Academic Press, p.137.

Thorson, J., 1978; Reconstruction of a wavefield from slant stacks, SEP-14, p.81.

SCHOLASTIC APTITUDE TEST SCORES

Students scoring	Total in U.S. 1972	Total in U.S. 1982	Applying to Stanford 1972	Applying to Stanford 1982	Stanford share 1972	Stanford share 1982
Verbal SAT 650, above	53,794	29,236	3,872	5,260	4%	7%
Verbal SAT 700, above	17,560	8,240	1,164	823	7	10
Verbal SAT 750, above	2,817	1,479	241	168	9	11
Math SAT 650, above	93,868	71,916	3,872	5,260	4	7
Math SAT 700, above	37,067	29,528	2,220	2,973	6	10
Math SAT 750, above	9,966	8,351	875	1,057	9	13

Verbal SAT	Number of applicants	Number offered admission	Math SAT	Number of applicants	Number offered admission
700-800	1,024	463	700-800	2,913	986
600-699	3,681	1,121	600-699	5,051	1,024
500-599	4,636	672	500-599	3,361	391
Below 500	3,252	212	Below 500	1,267	67

% college-bound H.S. seniors indicating intended field of study (from SAT data)

Year	Math	Physical Sciences	Engineering	English/Lit
1972	4	4	6	4
1973	3	4	6	4
1974	3	4	5	3
1975	2.4	2.8	6.7	2.4
1976	1.9	2.4	8.4	1.9
1977	1.7	2.3	8.8	1.8
1978	1.4	2.3	9.4	1.6
1979	1.2	2.2	10.1	1.6
1980	1.1	2.1	11.1	1.5
1981	1.1	2.0	11.8	1.4
1982	1.1	1.9	12.6	1.4

Total U.S. students scoring 750 + on achievement tests

Achievement Test	1972	1982
English Composition	2,918	1,302
Math 1	5,679	2,218
American History	1,757	719
French	1,884	839
Biology	1,690	681
Chemistry	3,285	1,558