CHAPTER 6

Pseudo-P and Pseudo-S Waves in a Hexagonally Anisotropic Earth

Reflection seismic experiments usually contain a predominance of unconverted compressional waves. However, three-component and shear-source experiments are becoming more important, and these must be analyzed using the elastic wave equation. Anisotropy is a large effect on shear wave profiles, so anisotropic propagation effects should be included in any serious attempt at downward continuing shear or converted wave fields.

The goal in this chapter will the development of a set of finite difference operators for downward continuation. The discussion will be less rigorous than that of chapter 4, the chapter that discussed downward continuation operators for acoustic waves. Attention is paid in this development to stability, compatibility with the acoustic operator in the isotropic limit, and faithfulness vis-a-vis the Christoffel relations. Little attention is paid to operator commutativity and proper modeling of wave amplitudes.

The eigenmodes of an anisotropic, elastic medium

An isotropic, elastic medium supports two shear waves and a compressional wave. Similarly, an anisotropic medium supports a horizontally polarized shear wave, a pseudo-shear wave, and a pseudo-compressional wave. Each of the three waves has a dispersion relation that can be translated into a wave equation. In this section, the notation required for a study of wave propagation in an anisotropic medium is introduced. The medium is assumed to be hexagonally anisotropic with a symmetry axis parallel to the z axis. The section ends with a derivation of the Christoffel equations.

Elastic wave propagation analysis is plagued by subscripts. This is because stress and strain are tensors of rank 2, and elastic stiffness is a tensor of rank 4. Following Auld, it is possible to reduce the number of subscripts by a factor of two by agreeing on the convention

$$xx \rightarrow 1 \quad yy \rightarrow 2 \quad zz \rightarrow 3$$

$$yz \rightarrow 4 \quad xz \rightarrow 5 \quad xy \rightarrow 6$$

for the subscripts. Repeated indices will, as is usual, imply a summation unless otherwise stated. The wave equation for the displacement field u, a function of the material stiffness c and density ρ, is given by

$$\nabla_{iJ} c_{JK} \nabla_{Kj} u_j = \rho \omega^2 u_i$$

in the reduced subscript scheme. The ∇ symbol stands for one of two matrices in this formalism depending on the order of its subscripts. In matrix notation, this equation takes the form

$$\begin{bmatrix} D_x^H & 0 & 0 & 0 & D_z^H & D_y^H \\ 0 & D_y^H & D_z^H & D_z^H & 0 & D_x^H \\ 0 & 0 & 0 & D_y^H & D_x^H & 0 \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} D_x & 0 & 0 \\ 0 & D_y & 0 \\ 0 & 0 & D_z \\ 0 & D_z & D_y \\ D_z & 0 & D_x \\ D_y & D_x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$= \rho \omega^2 \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

where $2c_{66} = c_{11} - c_{12}$.

Two simplifying assumptions will be made so that we may be able to derive three Christoffel equations. The first is that the experiment and medium are independent of y. The second is that the SH, pseudo-P, and pseudo-S waves are decoupled from one another. The justification for these two assumptions is that they are almost universally applied with some degree of success in the processing of seismic reflection data. The assumption of y-invariance decouples the SH wave from the other elastic waves that propagate in an anisotropic, elastic earth. With a little effort it is possible to show that the y-invariant wave equation for u is

$$\begin{bmatrix} D_x^H c_{11} D_x + D_z^H c_{44} D_z - \rho\omega^2 & 0 & D_x^H c_{13} D_z + D_z^H c_{44} D_x \\ 0 & D_z^H c_{44} D_z + D_x^H c_{66} D_x - \rho\omega^2 & 0 \\ D_z^H c_{13} D_x + D_x^H c_{44} D_z & 0 & D_z^H c_{33} D_z + D_x^H c_{44} D_x - \rho\omega^2 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

The decoupling assumption justifies the treatment of the material parameters in the matrix in equation (1) as if they were independent of position. In implementing the decoupling assumption, derivatives with respect to the material parameters in the matrix in equation (1) will be ignored. Later, the elastic parameters will be placed in positions with respect to the various derivatives in a way calculated to make the downward continuation equations in this chapter compatible with the development of the acoustic downward continuation equations of chapter 4. The wave equations for the three eigenmodes can be found by formally setting the determinant of the matrix equal to zero. This determinant can be factored into two pieces, equations (2) and (3),

$$D_z^H c_{44} D_z + D_x^H c_{66} D_x - \rho\omega^2 = 0 \qquad (2)$$

$$\det \begin{bmatrix} D_x^H c_{11} D_x + D_z^H c_{44} D_z - \rho\omega^2 & D_x^H c_{13} D_z + D_z^H c_{44} D_x \\ D_z^H c_{13} D_x + D_x^H c_{44} D_z & D_z^H c_{33} D_z + D_x^H c_{44} D_x - \rho\omega^2 \end{bmatrix} = 0 \qquad (3)$$

Equation (2) governs the propagation of the SH wave, the horizontally polarized shear mode The downward continuation equation for this mode can be derived by following the development of chapter 4. Similarly, equation (3) governs the joint propagation of pseudo-P and pseudo-S waves. The decoupling assumption implies that equation (3) can be factored into two equations with negligible error, one factor for each of the propagation modes. The factorization is performed by formally (ignoring restrictions on the commutativity of the operators) expanding the determinant in equation (3), replacing D_z^H with $-D_z$, and solving the resulting quartic for D_z^2. Saving the reader the grief associated with most of the worst of the algebra,

$$D_z^2 = \frac{\rho^2}{2c_{44}c_{33}}\left\{\frac{c_{11}c_{33}+c_{44}^2}{\rho^2}D_x^HD_x - \frac{c_{33}+c_{44}}{\rho}\omega^2 - \left|\frac{c_{13}}{\rho}D_x^H - \frac{c_{44}}{\rho}D_x\right|^2\right.$$
$$\pm\left[\left(\frac{c_{11}c_{33}+c_{44}^2}{\rho^2}D_x^HD_x - \frac{c_{33}+c_{44}}{\rho}\omega^2 - \left|\frac{c_{13}}{\rho}D_x^H - \frac{c_{44}}{\rho}D_x\right|^2\right)^2\right.$$
$$\left.\left. - \frac{4c_{44}c_{33}}{\rho^2}\left(\frac{c_{11}}{\rho}D_x^HD_x - \omega^2\right)\left(\frac{c_{44}}{\rho}D_x^HD_x - \omega^2\right)\right]^{1/2}\right\} \tag{4}$$

where the '+' and 's' signs should be chosen for pseudo-P and pseudo-S wave propagation, respectively. Differential equation (4) can be further simplified by using the equality $D_x^H = -D_x$ and ignoring derivatives of the elastic parameters. The result is a simpler differential equation

$$D_z^2 = \frac{\rho^2}{2c_{44}c_{33}}\left\{\frac{c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}}{\rho^2}D_x^HD_x - \frac{c_{33}+c_{44}}{\rho}\omega^2\right.$$
$$\pm\left[\left(\frac{c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}}{\rho^2}D_x^HD_x - \frac{c_{33}+c_{44}}{\rho}\omega^2\right)^2\right.$$
$$\left.\left.-4\frac{c_{44}c_{33}}{\rho^2}\left(\frac{c_{11}}{\rho}D_x^HD_x - \omega^2\right)\left(\frac{c_{44}}{\rho}D_x^HD_x - \omega^2\right)\right]^{1/2}\right\} \tag{5}$$

that lacks terms proportional to $D_x^H + D_x$. Differential equation (5) is still second order in z. As in the acoustic case, a one-way wave equation that is first order in depth is desirable. To get a partial differential equation with this form, take the square root of the operators both sides of equation (5). As in the acoustic case, there are many matrix square roots to choose from. Of these, only two square roots interesting, corresponding to the two equations for the downward continuation of upward and downward traveling wave fields. The equation for downward continuing upgoing pseudo-S or pseudo-P waves is

$$D_z = \frac{\rho}{\sqrt{2c_{44}c_{33}}}\left\{\frac{c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}}{\rho^2}D_x^HD_x-\frac{c_{33}+c_{44}}{\rho}\omega^2\right.$$

$$\pm\left[\left[\frac{c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}}{\rho^2}D_x^HD_x-\frac{c_{33}+c_{44}}{\rho}\omega^2\right]^2\right.$$

$$\left.\left.-4\frac{c_{44}c_{33}}{\rho^2}\left[\frac{c_{11}}{\rho}D_x^HD_x-\omega^2\right]\left[\frac{c_{44}}{\rho}D_x^HD_x-\omega^2\right]\right]^{1/2}\right\}^{1/2} \tag{6}$$

Equations for downwarding continuing downgoing waves can be had by substituting $-D_z$ for D_z in equation (6).

The differential equation for downward continuing waves in an anisotropic medium, equation (6), has two square roots that must be expanded. The trick is to do the expansion so that the acoustic one-way wave equation is obtained in the isotropic limit.

To get the isotropic limit, some restrictions need to be placed on the elastic constants. The constraints on the elastic constants for isotropy are that $c_{66} = c_{44}$, $c_{12} = c_{13}$, $c_{12} = c_{11}-2c_{44}$, and $c_{11} = c_{33}$. One result that follows immediately from the constraint equations is $2c_{11}c_{44} = c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}$. Plugging these relations into equation (6) yields two isotropic wave equations

$$D_z = \frac{\rho}{\sqrt{2c_{44}c_{33}}}\left\{\frac{2c_{11}c_{44}}{\rho^2}D_x^HD_x - \frac{c_{11}+c_{44}}{\rho}\omega^2\pm\left[\left[\frac{c_{11}-c_{44}}{\rho}\omega^2\right]^2\right]^{1/2}\right\}^{1/2}$$

where, in general $c_{11}\geq c_{44}$. With this last restriction, the inner square root and the square in its argument are inverse operations. Thus, the isotropic wave equations are

$$D_z = \frac{\rho}{\sqrt{2c_{44}c_{33}}}\left\{\frac{2c_{11}c_{44}}{\rho^2}D_x^HD_x - \frac{c_{11}+c_{44}}{\rho}\omega^2\pm\frac{c_{11}-c_{44}}{\rho}\omega^2\right\}^{1/2}$$

Some fortunate cancellations occur at this juncture, yielding a set of two isotropic wave equations. To get this cancellation, the sum of the ω^4 terms under the inner square root radical in equation (6) must be made to interact with the ω^2 outside that radical. This is the first important clue in the search for anisotropic finite difference operators that will be compatible with acoustic operators in the isotropic limit. A second is that the sum of the terms proportional to $D_x^H D_x \omega^2$ under the inner radical must vanish identically in the isotropic limit. The same conclusion holds for the sum of terms proportional to $(D_x^H D_x)^2$ under the inner square root radical. The result is a set of two differential equations

$$D_z = \frac{\rho}{\sqrt{2c_{44}c_{11}}}\left\{\frac{2c_{11}c_{44}}{\rho^2} D_x^H D_x - \frac{2c_{44}}{\rho}\omega^2\right\}^{1/2} \qquad (+\ sign)$$

$$D_z = \frac{\rho}{\sqrt{2c_{44}c_{11}}}\left\{\frac{2c_{11}c_{44}}{\rho^2} D_x^H D_x - \frac{2c_{11}}{\rho}\omega^2\right\}^{1/2} \qquad (-\ sign)$$

where it is now seen that the '+' and '-' signs should be chosen for the downward continuation of pseudo-P and pseudo-S waves, respectively. Fourth, an indication of where the elastic constants should appear with respect to the square root radicals and derivatives is found in the acoustic equation. That equation sandwiches $D_x^H D_x$ between two velocities and surrounds its square root with square roots of the acoustic slownesses. Fifth, the square roots must have two arguments that commute with one another. Finally, the operators involving $D_x^H D_x$ should be positive semi-definite forms. We begin by symmetrizing

$$D_z = \left[\frac{\rho}{\sqrt{2c_{44}c_{33}}}\right]^{1/2}\left\{w_1 D_x^H D_x w_1 + \frac{c_{33}+c_{44}}{\rho}(i\omega)^2\right.$$

$$\left.\pm\left[\left[\frac{c_{33}-c_{44}}{\rho}\right]^2(i\omega)^4 + \omega^2 w_2 D_x^H D_x w_2 + w_3 D_x^H D_x D_x^H D_x w_3\right]^{1/2}\right\}^{1/2}\left[\frac{\rho}{\sqrt{2c_{44}c_{33}}}\right]^{1/2}$$

$$w_1 = \left[\frac{c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}}{\rho^2}\right]^{1/2}$$

$$w_2 = \left[\frac{4c_{44}c_{33}(c_{11}+c_{44})-2(c_{33}+c_{44})(c_{11}c_{33}-c_{13}^2-2c_{13}c_{44})}{\rho^3}\right]^{1/2}$$

$$w_3 = \left[\frac{(c_{11}c_{33}-c_{13}^2-2c_{13}c_{44})^2-4c_{11}c_{33}c_{44}^2}{\rho^4}\right]^{1/2}$$

The above expression for D_z is still not acceptable since the operators under the various square root radicals do not commute with one another. To make downward continuation operators for which the necessary commutivity properties hold, it will prove convenient to pick a sign and specialize the discussion to cover pseudo-S waves alone.

Downward continuation of pseudo-S waves

To get a downward continuation operator for an upgoing pseudo-S wave from the last set of equations, just pick the '-' sign. Once this is done, the two square roots can be consistently defined. The solution to the expansion problem lies in rewriting the equation so that the square roots' arguments commute with one another.

We begin with the inner square root by expanding it in a continued fraction. Only the first approximant is kept in a use of the familiar 15-degree approximation. Higher order approximants do not lead to theoretical difficulties, but do cloud the algebra. A necessary first step is to pick the partial numerators and partial denominators of the expansion. Remembering the need to make the terms that are independent of $D_x^H D_x$ interact, it is natural to single out the $(i\omega)^4$ term as the partial denominator of the expansion in the inner square root. Since it is also desirable that this operator commute with the other terms beneath the inner radical, the multiplier of $(i\omega)^4$ should be set equal to one. Factoring $(c_{33}-c_{44})^2/\rho^2$ symmetrically and applying the 15-degree approximation yields

$$D_z = \left(\frac{\rho}{\sqrt{2c_{44}c_{33}}}\right)^{1/2} \left\{ w_1 D_x^H D_x w_1 + \frac{c_{33}+c_{44}}{\rho}(i\omega)^2 - \frac{c_{33}-c_{44}}{\rho}(i\omega)^2 + \right.$$

$$\left. -\left(\frac{c_{33}-c_{44}}{\rho}\right)^{1/2} \frac{\omega^2 w_2 D_x^H D_x w_2 + w_3 D_x^H D_x D_x^H D_x w_3}{2(i\omega)^2} \left(\frac{c_{33}-c_{44}}{\rho}\right)^{1/2} \right\}^{1/2} \left(\frac{\rho}{\sqrt{2c_{44}c_{33}}}\right)^{1/2}$$

On combining the $(i\omega)^2$ terms, the c_{33} disappears. The $(i\omega)^2$ term, therefore, has a coefficient equal to $2c_{44}/\rho$. If this coefficient is factored outside the square root in the center, then two arguments of this square root will commute with one another. Just before the expansion

$$D_z = \left[\frac{\rho}{c_{33}}\right]^{1/4} \left\{(i\omega)^2 + \left[\frac{\rho}{2c_{44}}\right]^{1/2} w_1 D_x^H D_x w_1 \left[\frac{\rho}{2c_{44}}\right] + A\right\}^{1/2} \left[\frac{\rho}{c_{33}}\right]^{1/4}$$

$$A = \left[\frac{\rho}{2c_{44}}\right]^{1/2} \left[\frac{c_{33}-c_{44}}{\rho} \frac{(\omega^2 w_2 D_x^H D_x w_2 + w_3 D_x^H D_x D_x^H D_x w_3)}{2(\omega)^2} \frac{c_{33}-c_{44}}{\rho}\right] \left[\frac{\rho}{2c_{44}}\right]^{1/2}$$

Again expanding the square root, this time to infinite order, the partial numerator is a complicated expression. In contrast, the partial denominator is equal to $2i\omega$, as it is in the acoustic case. If an approximation higher than the 15-degree fraction were to be used in the expansion for the inner square root of the dispersion relation, then the partial numerators here would be even more complex.

$$D_z = (i\omega)\left[\frac{\rho}{c_{33}}\right]^{1/2} + \left[\frac{\rho}{c_{33}}\right]^{1/4} \left[\cfrac{w_1 D_x^H D_x w_1 + A}{2i\omega + \cfrac{w_1 D_x^H D_x w_1 + A}{2i\omega + \cfrac{w_1 D_x^H D_x w_1 + A}{2i\omega + \cdots}}}\right] \left[\frac{\rho}{c_{33}}\right]^{1/4}$$