CHAPTER 5

Magic Numbers

The last chapter introduced a few parameters for use in finite difference migration codes. The most important of these is β, which modifies the denominator of the discrete, rational representation of the second difference operator. Less important is the complex number ψ, used to adjust the fit the dispersion relations of the 45-degree and higher order equations to the dispersion for the one-way wave equation. Finally, there are the $a_j^{(k)}$'s and $b_j^{(k)}$'s needed to carry out Ma's method. This chapter considers methods for determining appropriate values for these parameters.

Dispersion relations, dip filtering, and ψ

Finite difference migration algorithms use rational approximations to the square root that appears in the dispersion relation for the one-way wave equation. As can be seen in figure 1, the one-way wave equation (in the absence of dissipation) partitions the (k_x, ω) plane into two regions. The most interesting region of the plane is where the square root of the one-way wave equation has a real and positive argument. The other region is one where the square root's argument is real and negative. Such regions correspond to evanescent waves that decay and do not propagate for long distances in the earth. The rational operators have a slightly different behavior in both the propagating and the evanescent regions of the wavenumber plane.

Within the evanescent region, the rational approximations all support propagating wave modes. Their behavior in this zone is distinctly unphysical and causes unsightly artifacts. One purpose of ψ is to dissipate energy propagated with wavenumbers in the evanescent zone.

At each point in the propagating region the dispersion relation for the wave equation makes it possible to calculate a vertical wavenumber. Similarly, the dispersion relations for the rational approximations to the wave equation make it possible to calculate a vertical wavenumber estimate at each point in the propagating region. While both the wavenumber and its estimates will be real numbers, they will, in general, differ. The magnitude of the difference is zero

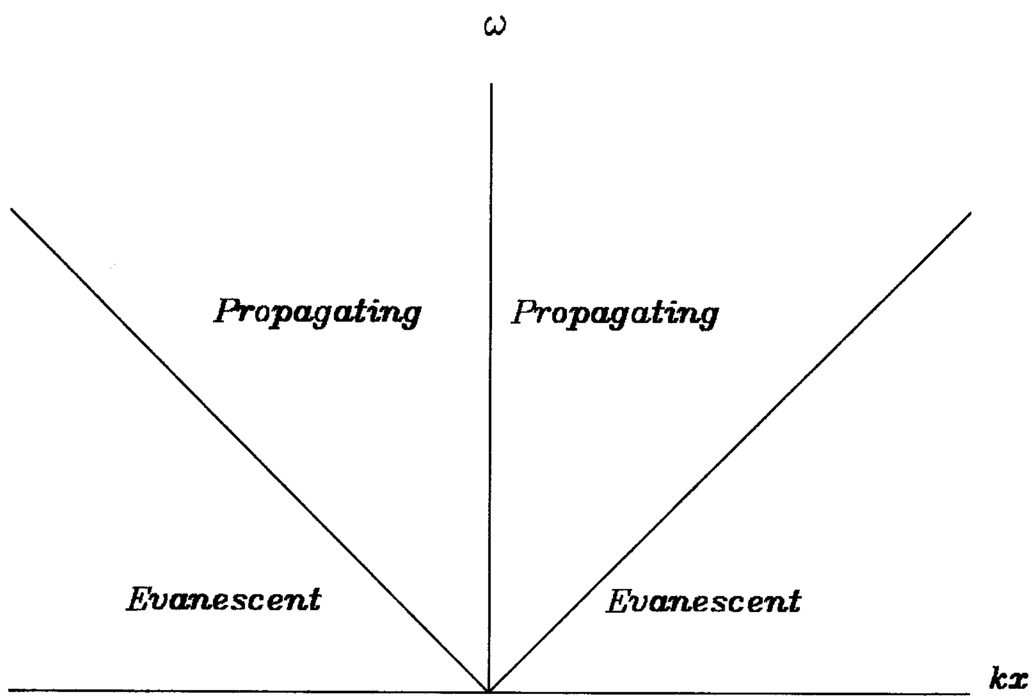

FIGURE 5.1 Evanescent and propagating zones. Wave propagation occurs for small values of k_x in the (k_x, k_z) plane. Plane waves with sufficiently large values of $|k_x|$ are attenuated instead of propagated. The boundary between the propagating and evanescent regions consists of straight lines with slopes $\pm V/\omega$, where V is the acoustic velocity of the medium and ω is the temporal frequency.

along the k_z axis, and usually increases dramatically with increasing horizontal wavenumbers. The differences between the vertical wavenumbers and their estimates may be small, but they remain important. This is because small changes in wavenumber estimates translate into systematic errors in the location estimates of geological structures in the subsurface. These systematic errors are larger for events with big dips than for events with small ones. Another

Order(j)	ψ_i	ψ_r	E_j
1	0.00	0.7300	1.556e-05
	0.01	0.7302	1.584e-05
	0.02	0.7308	1.667e-05
	0.03	0.7318	1.806e-05
	0.04	0.7332	1.998e-05
	0.05	0.7349	2.242e-05
	0.06	0.7370	2.536e-05
	0.07	0.7395	2.878e-05
	0.08	0.7424	3.265e-05
	0.09	0.7455	3.694e-05
	0.10	0.7490	4.162e-05
2	0.00	0.6553	4.701e-07
	0.01	0.6556	4.878e-07
	0.02	0.6566	5.405e-07
	0.03	0.6582	6.269e-07
	0.04	0.6605	7.451e-07
	0.05	0.6633	8.929e-07
	0.06	0.6667	1.067e-06
	0.07	0.6706	1.265e-06
	0.08	0.6749	1.483e-06
	0.09	0.6797	1.718e-06
	0.10	0.6850	1.967e-06
3	0.00	0.6203	3.477e-08
	0.01	0.6208	3.736e-08
	0.02	0.6221	4.502e-08
	0.03	0.6242	5.746e-08
	0.04	0.6271	7.421e-08
	0.05	0.6308	9.473e-08
	0.06	0.6350	1.184e-07
	0.07	0.6399	1.445e-07
	0.08	0.6453	1.726e-07
	0.09	0.6512	2.021e-07
	0.10	0.6574	2.325e-07

TABLE 5.1. Optimum values of ψ_r. Best values for ψ_r for various j, ψ_i combinations. Weighting by a factor of $(1 + q)/2$ in the propagating region, where $-1 < q < 0$, is assumed.

reason for introducing ψ is to achieve as accurate an approximation over as large a dip range as possible.

To begin the search for ψ, dispersion relations for the differential equations involved and some measure of distance between them are needed. The dispersion relations are easily derived from the wave equation and the recursion for the approximants to the wave equation operator. Let c_0 and q stand for vk_z/ω and $-(vk_x/\omega)^2$, respectively. The variable c_0 is the cosine of the propagation angle of a plane wave satisfying the one-way wave equation. Similarly, q is the negative square of the sine of the propagation angle of such a wave. The negative square of the sine is treated with, instead of the sine, to simplify the algebra that follows. Finally, let c_j stand for $vk_{z,j}/\omega$, where $k_{z,j}$ is the estimate of the vertical wavenumber obtained from the dispersion relation of the j-th rational approximation to the wave equation. The dispersion relations are functions of these variables and are given by

$$c_0 = \left(1 + q\right)^{1/2}$$

$$\text{Re}\, c_1 = 1 + \frac{2q\psi_r(4\psi_r + q) + 8q\psi_i^2}{(4\psi_r + q)^2 + 16\psi_i^2} \tag{1}$$

$$\text{Im}\, c_1 = \frac{2q^2\psi_i}{(4\psi_r + q)^2 + 16\psi_i^2} \tag{2}$$

$$\text{Re}\, c_j = 1 + \frac{q(1+\text{Re}\, c_{j-1})[2(1+\text{Re}\, c_{j-1})+q] + 2q(\text{Im}\, c_{j-1})^2}{[2(1+\text{Re}\, c_{j-1})+q]^2 + 4(\text{Im}\, c_{j-1})^2} \tag{3}$$

$$\text{Im}\, c_j = \frac{q^2\text{Im}\, c_{j-1}}{[2(1+\text{Re}\, c_{j-1})+q]^2 + 4(\text{Im}\, c_{j-1})^2} \tag{4}$$

where $\psi = \psi_r + i\,\psi_i$. The variables Re c_1 and Re c_j are estimates for c_0, while Im c_1 and Im c_j should be close to zero. The latter pair are responsible for dip filtering in the evanescent region where $q < -1$. A rough size estimate for ψ_i can be made by considering a monochromatic wave with temporal frequency ω traveling with a large stepout through a material with thickness Δz and acoustic velocity V. Such a wave is attenuated by a factor of $\exp[-2\psi_i \omega V^{-1} \Delta z]$. Suppose that a desirable attenuation rate for the monochromatic wave is 10 per cent per z step. If $\omega = 300/s$, $V = 2000$ m/s, and $\Delta z = 30$ m, then ψ_i should be nearly equal to 0.01. A good way to pick ψ_i is to assign it a value at infinite "propagation angle" using the limiting relations

$$\lim_{q \to -\infty} \text{Im } c_1 = \lim_{q \to -\infty} \text{Im } c_j = 2\psi_i \tag{5}$$

A best choice for ψ_r can be made by minimizing the propagating region deviations of Re c_j and Im c_j from c_0 and zero, respectively. One norm that suggests itself is

$$E_j = \int_{-1}^{0} dq \left[W_r(q)(\text{Re } c_j - c_0)^2 + W_i(q)(\text{Im } c_j)^2 \right] \tag{6}$$

The norm in (6) should be minimized by variation of ψ_r. Newton's method is one method of doing the minimization. An implementation of Newton's method requires first and second derivatives of E_j with respect to ψ_r. These two derivatives can be expressed as functions of the first two derivatives of Re c_j and Im c_j with respect to ψ_r. The derivatives of Re c_j and Im c_j can be computed using the recursions found in the appendix. The recursions are direct consequences of equations (1) through (4). The equations for the derivatives of E_j as functions of the derivatives of Re c_j and Im c_j are

$$\frac{\partial E_j}{\partial \psi_r} = 2\int_{-1}^{0} dq \left[W_r(\text{Re } c_j - c_0) \frac{\partial \text{Re } c_j}{\partial \psi_r} + W_i \text{Im } c_j \frac{\partial \text{Im } c_j}{\partial \psi_r} \right]$$

$$\frac{\partial^2 E_j}{\partial \psi_r^2} = 2\int_{-1}^{0} dq \left\{ W_r \left[\left(\frac{\partial \text{Re } c_j}{\partial \psi_r} \right)^2 + (\text{Re } c_j - c_0) \frac{\partial^2 \text{Re } c_j}{\partial \psi_r^2} \right] + W_i \left[\left(\frac{\partial \text{Im } c_j}{\partial \psi_r} \right)^2 + \text{Im } c_j \frac{\partial^2 \text{Im } c_j}{\partial \psi_r^2} \right] \right\}$$

These derivatives are employed to update an incorrect guess for the value of ψ_r that minimizes the norm E_j according to the scheme

$$\psi_r \leftarrow \psi_r - \left(\frac{\partial^2 E_j}{\partial \psi_r^2} \right)^{-1} \frac{\partial E_j}{\partial \psi_r}$$

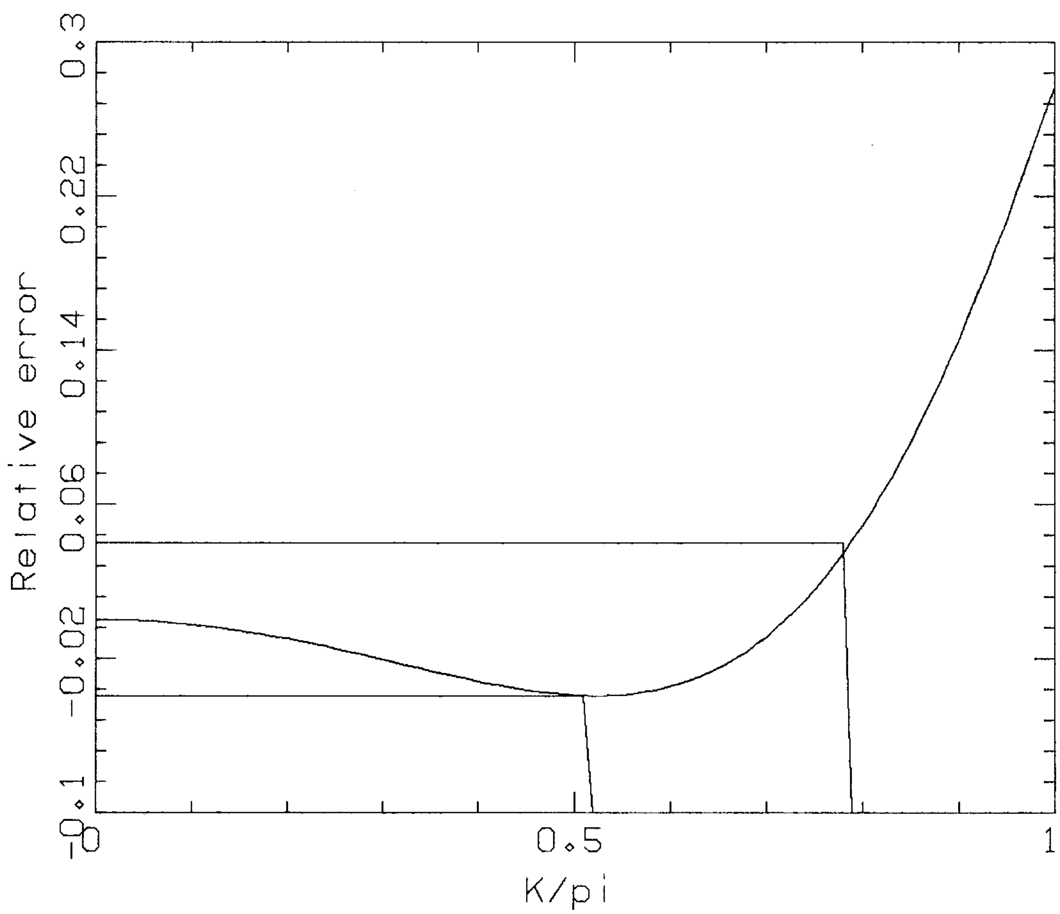

FIGURE 5.2. Second derivative error. When β lies between 1/12 and 1/4, the relative deviation of the discrete approximation to the continuous second derivative operator has a minimum at K_m, the point where the relative error is $-E^*_\beta$. K_β is the argument where the relative error curve is equal to $+E^*_\beta$.

Table 1 contains a list of optimum ψ_r's for various combinations of orders j and dip filtering parameters ψ_i. The choice of weights W_r and W_i is more or less arbitrary. The weights that were used to make table 1 were $W_r = W_i = 1/2(1+q)$ over the interval $-1 < q < 0$. Newton's method converged quickly with this weighting scheme.

β	K_m	K_β	100 E_β^*
1/12	0.0	0.0	0.0
0.0834	0.08954	0.1389	2.666e-05
0.0840	0.2821	0.4379	2.658e-03
0.0850	0.4445	0.6886	1.654e-02
0.0860	0.5605	0.8668	4.215e-02
0.0870	0.6547	1.0110	7.935e-02
0.0880	0.7364	1.1350	1.280e-01
0.0890	0.8086	1.2450	1.880e-01
0.0900	0.8740	1.344	2.592e-01
0.0950	1.138	1.738	7.805e-01
0.1000	1.339	2.033	1.571
0.1050	1.504	2.273	2.629
0.1100	1.644	2.475	3.954
0.1150	1.767	2.651	5.553
0.1200	1.876	2.807	7.438
0.1250	1.975	2.947	9.623
0.1300	2.064	3.076	12.13
0.1310	2.081	3.100	12.67
0.1320	2.098	3.125	13.23
0.13268	2.112	π	13.72

TABLE 5.2. Optimum values for β. Value of beta between 1/12 and 0.13286 solve minimax relative error approximation problems over an interval for a discrete second difference operator. The limits of the interval are zero and K_β. The error is maximum at K_β and at K_m, where $0 \le K_m \le K_\beta$. The value of the error at this maximum is denoted by E_β^*.

Second derivative approximations and β

Perhaps the most important of the migration parameters is β, used in discrete representations of the second derivative operator. As a function of the negative of the discrete second-differencing matrix T, the approximate second derivative with respect to x has the form $-(\Delta x)^{-2} T/(I - \beta T)$. The parameter β should be chosen to get as good a fit as possible over as wide a range of k_x's as possible.

To get a comparison of the discrete and continuous second derivative operators for a wavenumber β, the frequency domain analogues of the two operators must be found. The continuous second derivative operator is multiplicative in the frequency domain, with the factor equal to $-k_x^2$. The discrete operator is multiplicative, too. By taking a discrete Fourier transform, the multiplicative factor for the discrete second differencing operator is

$$\frac{-4}{(\Delta x)^2} \frac{\sin^2(k_x \Delta x / 2)}{1 - 4\beta \sin^2(k_x \Delta x / 2)}$$

valid for $k_x \Delta x$ between $-\pi$ and π. Thus, the relative error between the discrete and continuous derivatives is equal to

$$E_\beta = 1 - \frac{4\sin^2(K/2)}{K^2[1 - 4\beta\sin^2(K/2)]}$$

where $K = k_x \Delta x$. To keep the denominator of this rational form positive it is necessary that β be less than 1/4.

Consider the error for K between D.C. and Nyquist, β between zero and 1/4. When β is less than 1/12, the relative error curve is increasing on the interval between zero and π. The slope at D.C., however, is decreasing with increasing β. When β is equal to 1/12, the slope of the relative error curve at the origin is equal to zero. This implies that if a data set has a small bandwidth, then the best choice for β is 1/12. For larger values of β the relative error curve has a maximum at π and a minimum at some point between zero and π. In figure 2 this minimum is denoted K_m, the value of the relative error at the minimum by $-E_\beta^*$, and the point where the relative error is equal to $+E_\beta^*$ is called K_β. Obviously, β solves the minimax relative error problem for second difference approximation over the interval $0 < K < K_\beta$. Values of K_m, K_β, and E_β^* are tabulated opposite values of β in table 2. Only values of β between 1/12, where K_β is equal to zero, and 0.13286, where K_β is equal to π are plotted.

If a migration algorithm works in the (ω, x) domain, then the minimax β's can be used to get an ω dependent second difference operator. When the temporal frequency of a wavefield is ω and the smallest acoustic velocity is V, then the largest k_x for which propagation will occur is equal to ωV^{-1}. The parameter beta should then be chosen so that the minimax interval has endpoints at zero and min($\omega V^{-1}\Delta x,\pi$). By least squares fitting of the data tabulated in table 2,

$$\beta(K_\beta) \approx 6.550e-03\, K^2 - 5.888e-03\, K + 8.658e-02$$

$$E_\beta^*(K_\beta) \approx \exp[-1.425\, K^2 + 8.305\, K - 14.45]$$

where $K = k_x \Delta x$. These relations hold reasonably well for K between 0.08954 and π. Between zero and 0.08954, the linear relationship

$$\beta(K_\beta) \approx \frac{1}{12} + \frac{0.086105 - 1/12}{0.08954} K_\beta$$

seems reasonable. This equation preserves continuity at $K_\beta = 0.08954$ and equals 1/12 at $K_\beta = 0$.

Coefficients for Ma's methods

Ma's method is implemented by performing partial fractions on an approximant of the continued fraction for the square root in the one-way wave equation. The differential equation that results can then be split into its component pieces. In the process a set of parameters, $a_j^{(k)}$ and $b_j^{(k)}$ $(1 \leq j \leq k)$, are generated.

A continued fraction that is useful when considering the application of Ma's method to the one-way wave equation has approximants that obey a first-order recursion

$$R_1 = \frac{2\psi}{4\psi + S}$$

$$R_{k+1} = \cfrac{1}{2 + \cfrac{S}{2 + SR_k}}$$

$$S = (i\omega - \varepsilon)^{-2} \kappa^{1/2} D_x^H \frac{1}{\rho} D_x \kappa^{1/2}$$

The numerators and denominators of R_k have fraction bars. A first step towards getting a partial fraction expansion is to get rid of the fraction bars. This can be done by considering the relations

$$R_k = \frac{\sum_{m=1}^{k} c_m^{(k)} S^{m-1}}{\sum_{m=1}^{k} d_m^{(k)} S^{m-1} + S^k}$$

$$c_1^{(1)} = 2\psi$$

$$d_1^{(1)} = 4\psi$$

$$c_1^{(k+1)} = 2d_1^{(k)}$$

$$c_j^{(k+1)} = c_{j-1}^{(k)} + 2d_j^{(k)} \qquad 2 \le j \le k$$

$$c_{k+1}^{(k+1)} = c_k^{(k)} + 2$$

$$d_1^{(k+1)} = 4d_1^{(k)}$$

$$d_j^{(k+1)} = 4d_j^{(k)} + d_{j-1}^{(k)} + 2c_{j-1}^{(k)} \qquad 2 \le j \le k$$

$$d_{k+1}^{(k+1)} = 4 + d_k^{(k)} + 2c_k^{(k)}$$

Once the numerator and denominator of the approximant of order k have been cleared, the k roots of the denominator must be found. The denominator polynomial can be factored analytically when $k < 5$. Higher order polynomials can only be factored by numerical methods. Fortunately, this has only theoretical significance for users of finite difference migration algorithms. The numerators of the partial fraction terms can be found by solving an appropriate system of linear equations.

Appendix - The calculation of derivatives with respect to ψ_r

Given values for ψ_r, ψ_i, k, and q, some method for calculating the derivatives of Re c_j, Im c_j, and E_j with respect to ψ_r is needed. The calculation is analytic for $k = 1$ and recursive for all others. Both the analytic and recursive formulae are based on equations (1) through (4).

$$DEN_1 = (4\psi_r + q)^2 + 16\psi_i^2$$

$$\frac{\partial \text{Re } c_1}{\partial \psi_r} = \frac{2q(8\psi_r + q)}{DEN_1} - \frac{16q[\psi_r(4\psi_r+q)^2 + 4\psi_i^2(4\psi_r+q)]}{DEN_1^2}$$

$$\frac{\partial^2 \text{Re } c_1}{\partial \psi_r^2} = \frac{16q}{DEN_1} - \frac{32q(4\psi_r+q)^2+192q\psi_r(4\psi_r+q)+256q\psi_i^2}{DEN_1^2} + \frac{256q[\psi_r(4\psi_r+q)^3+4\psi_i^2(4\psi_r+q)^2]}{DEN_1^3}$$

$$\frac{\partial \text{Im } c_1}{\partial \psi_r} = -\frac{16q^2\psi_i(4\psi_r+q)}{DEN_1^2}$$

$$\frac{\partial^2 \text{Im } c_1}{\partial \psi_r^2} = -\frac{64q^2\psi_i}{DEN_1^2} + \frac{256q^2\psi_i(4\psi_r+q)^2}{DEN_1^3}$$

The derivatives of Re c_j and Im c_j for $j > 1$ have even more complex analytic expressions. It is profitable to consider a recursive scheme for computing the derivatives of Re c_j and Im c_j as functions of the derivatives of Re c_{j-1} and Im c_{j-1} and the values of 1 + Re c_{j-1} and Im c_{j-1}.

$$x_j = 1+\text{Re } c_j \qquad y_j = \text{Im } c_j$$

$$DEN_j = (2x_{j-1}+q)^2 + 4y_{j-1}^2$$

$$A = \frac{2qx_{j-1}+q(2x_{j-1}+q)}{DEN_j} - \frac{4q[x_{j-1}(2x_{j-1}+q)^2+2y_{j-1}^2(2x_{j-1}+q)}{DEN_j^2}$$

$$B = \frac{4qy_{j-1}}{DEN_j} - \frac{8qy_{j-1}[x_{j-1}(2x_{j-1}+q)+2y_{j-1}^2]}{DEN_j^2}$$

$$C = \frac{4q}{DEN_j} - \frac{24qx_{j-1}(2x_{j-1}+q)+8q(2x_{j-1}+q)^2+16qy_{j-1}^2}{DEN_j^2} + \frac{32q[x_{j-1}(2x_{j-1}+q)^3+2y_{j-1}^2(2x_{j-1}+q)^2}{DEN_j^3}$$

$$D = \frac{4q}{DEN_j} - \frac{80qy_{j-1}^2+8qx_{j-1}(2x_{j-1}+q)}{DEN_j^2} + \frac{128qy_{j-1}^2[x_{j-1}(2x_{j-1}+q)+2y_{j-1}^2]}{DEN_j^3}$$

$$E = -\frac{32qx_{j-1}y_{j-1}+48qy_{j-1}(2x_{j-1}+q)}{DEN_j^2} + \frac{128qy_{j-1}[x_{j-1}(2x_{j-1}+q)^2+2y_{j-1}^2(2x_{j-1}+q)]}{DEN_j^3}$$

$$F = \frac{-4q^2y_{j-1}(2x_{j-1}+q)}{DEN_j^2}$$

$$G = \frac{q^2}{DEN_j} - \frac{8q^2 y_{j-1}^2}{DEN_j^2}$$

$$H = -\frac{8q^2 y_{j-1}}{DEN_j^2} + \frac{32q^2 y_{j-1}(2x_{j-1}+q)^2}{DEN_j^3}$$

$$I = -\frac{24q^2 y_{j-1}}{DEN_j^2} + \frac{128q^2 y_{j-1}^3}{DEN_j^3}$$

$$J = -\frac{8q^2(2x_{j-1}+q)}{DEN_j^2} + \frac{128q^2 y_{j-1}^2(2x_{j-1}+q)}{DEN_j^3}$$

$$\frac{\partial \mathrm{Re}\ c_j}{\partial \psi_r} = A\ \frac{\partial \mathrm{Re}\ c_{j-1}}{\partial \psi_r} + B\ \frac{\partial \mathrm{Im}\ c_{j-1}}{\partial \psi_r}$$

$$\frac{\partial^2 \mathrm{Re}\ c_j}{\partial \psi_r^2} = A\frac{\partial^2 \mathrm{Re}\ c_{j-1}}{\partial \psi_r^2} + B\frac{\partial^2 \mathrm{Im}\ c_{j-1}}{\partial \psi_r^2} + C\left(\frac{\partial \mathrm{Re} c_{j-1}}{\partial \psi_r}\right)^2 + D\left(\frac{\partial \mathrm{Im} c_{j-1}}{\partial \psi_r}\right)^2 +$$

$$E\frac{\partial \mathrm{Re} c_{j-1}}{\partial \psi_r}\ \frac{\partial \mathrm{Im} c_{j-1}}{\partial \psi_r}$$

$$\frac{\partial \mathrm{Im}\ c_j}{\partial \psi_r} = F\frac{\partial \mathrm{Re}\ c_{j-1}}{\partial \psi_r} + G\ \frac{\partial \mathrm{Im}\ c_{j-1}}{\partial \psi_r}$$

$$\frac{\partial^2 \mathrm{Im}\ c_j}{\partial \psi_r^2} = F\frac{\partial^2 \mathrm{Re}\ c_{j-1}}{\partial \psi_r^2} + G\frac{\partial^2 \mathrm{Im}\ c_{j-1}}{\partial \psi_r^2} + H\left(\frac{\partial \mathrm{Re} c_{j-1}}{\partial \psi_r}\right)^2 + I\left(\frac{\partial \mathrm{Im} c_{j-1}}{\partial \psi_r}\right)^2 +$$

$$J\frac{\partial \mathrm{Re} c_{j-1}}{\partial \psi_r}\ \frac{\partial \mathrm{Im} c_{j-1}}{\partial \psi_r}$$