CHAPTER 4

High Order Finite Difference Migration

Offset angles in a shot profile can become high even when the dip of the subsurface reflectors is small. If the Cartesian method for the migration of profiles is to yield a high quality image of the subsurface, then the migration scheme used must be able to migrate large offset angles. If a finite difference migration program is used, then the high offset angles require the use of high order migration operators.

Migration operators of arbitrary order can be built up from operators of lower order with a recursion. This chapter's first section introduces the wave equation for a pressure wave field. Four one-way wave equations can be extracted from the two-way wave equation for pressure, each propagating waves in a different vertical or temporal direction. The extraction uses a set of operators with well-defined causality. The second section of this chapter discusses derivatives and causality, while the one-way wave equation for the migration of an up-going wave field is derived in a third. The four sections that follow discuss computer implementations of the migration equation, building on a scheme first proposed by Ma (1982). The chapter concludes with displays of impulse responses and an analysis of the stability of finite difference downward continuation.

The pressure wave equation and acoustic media

One-way wave equations are the foundation of finite difference migration procedures. Finite differencing is done in a discrete world and requires the solution of a banded system of linear equations. Before obtaining the matrix coefficients, the proper one-way wave equation for migrating up-going waves will be derived.

The starting point is the wave equation for a pressure wave field $P(x,z,t)$ in an acoustic medium. The medium is characterized by its density $\rho(x,z)$ and bulk modulus $\kappa(x,z,t)$. These two functions define the medium's acoustic velocity and slowness, denoted by $V(x,z,t)$ and $\Lambda(x,z,t)$, respectively. The time dependence in V, Λ, and κ can be made to account for the visco-acoustic effects of dissipative wave equations. For causality, the reciprocal of κ will have to vanish for $t < 0$. The wave equation for P,

$$\frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\frac{1}{\rho}\frac{\partial P}{\partial x} - \int_{-\infty}^{t} dt' \frac{1}{\kappa(x,z,t-t')}\frac{\partial^2 P(x,z,t')}{\partial t'^2} = 0 \qquad (1)$$

governs the propagation of acoustic waves when combined with appropriate boundary conditions.

Second order boundary conditions must accompany equation (1) for it to have a unique solution. One such boundary condition is obtained by specifying values for P and its normal derivative on the surface $z = 0$ and by requiring that P and its derivatives vanish when either z or $|x| \to \infty$. Unfortunately, insufficient information is available for application of this boundary condition. The wave field $P(x,z,t)$ is known at $z = 0$, but the normal derivative of $P(x,z,t)$ on $z = 0$ is unmeasured. It is expedient, therefore, to consider one-way wave equations that support propagation in one vertical direction. Consideration of one-way wave equations will involve assigning causality to the derivatives rearrangement of the wave equation

Derivatives and causality

The derivatives in equation (1), will be handled first. A one-way wave equation has a causal character. This character must be respected by the derivatives of the differential equation. Thus, more care must be taken in defining a derivative than is taken in elementary calculus texts. In such texts, it is taught that the derivative of a function $f(t)$, denoted by $D_t^0 f(t)$ is defined when both

$$D_t f(t) = \lim_{\Delta t \to 0+} \frac{f(t+\Delta t) - f(t)}{\Delta t} \quad \text{and} \quad -D_t^H f(t) = \lim_{\Delta t \to 0+} \frac{f(t) - f(t-\Delta t)}{\Delta t}$$

exist and are equal. Here, $D_t^0 f(t) = D_t f(t) = -D_t^H f(t)$. This definition really involves three kinds of derivative operators: anti-causal derivatives like $-D_t^H$, causal derivatives like D_t, and derivatives like D_t^0 that exist when both $-D_t^H$ and D_t exist and are equal.

The one-way wave equation is peculiar in that derivatives of the D_t^0 type are not relevant. The other two types of derivatives, the causal operators (like D_t) and the anti-causal operators (like $-D_t^H$), are useful. Defining positive increments Δx, Δz, and Δt, six partial derivatives can be defined by the six equations

$$D_x f(x,z,t) = \frac{f(x,z,t) - f(x-\Delta x,z,t)}{\Delta x}, \quad -D_x^H f(x,z,t) = \frac{f(x+\Delta x,z,t) - f(x,z,t)}{\Delta x}$$

$$D_z f(x,z,t) = \frac{f(x,z,t) - f(x,z-\Delta z,t)}{\Delta z}, \quad -D_z^H f(x,z,t) = \frac{f(x,z+\Delta z,t) - f(x,z,t)}{\Delta z}$$

$$D_t f(x,z,t) = \frac{f(x,z,t) - f(x,z,t-\Delta t)}{\Delta t}, \quad -D_t^H f(x,z,t) = \frac{f(x,z,t+\Delta t) - f(x,z,t)}{\Delta t}$$

where limiting processes are performed if necessary. The H superscript denotes Hermitian conjugation. To see why this should be so, consider the case of x-differentiation in a discrete space consisting of an infinite set of evenly spaced grid points. In this space there are no boundaries to introduce anomalous values into the differencing matrices. D_x, for instance, is a matrix operator with a diagonal of $1/\Delta x$'s and a subdiagonal of $-1/\Delta x$'s. The negative of the transpose of D_x is a matrix that has a diagonal of $-1/\Delta x$'s and a superdiagonal of $1/\Delta x$'s. Thus, D_x^H has the form required for an anti-causal derivative.

The x derivatives discussed above give a consistent estimate of the derivative as Δx approaches zero. No notational distinction is made between operations in a discrete world and differential operations in a continuum. Other discrete operators with well-defined causality have the proper limit as Δx approaches zero, however. Unless explicitly indicated, the notation used in this thesis will not distinguish between continuous partial derivative operators and the various discrete differential operators that happen to have this continuous operator as a limiting value for small discretization rates. This practice reduces the number of symbols and simplifies most of the functional analysis. For example, it will turn out to be useful to consider an x derivative of the form

$$D_x f(x,z,t) \;=\; \frac{1}{\Delta x} B \left(I - \frac{1-2\beta+(1-4\beta)^{1/2}}{1+(1-4\beta)^{1/2}} B \right)^{-1} f(x,z,t)$$

where B is a causal operator and β is a real and positive constant that is less than 1/4. The constant beta is chosen to increase the bandwidth of the approximation to the continuous derivative. The operator B has a matrix representation that has a diagonal of 1's and a subdiagonal of -1's. Thus, the matrix representation of B is

$$B \;=\; \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & & \cdots & & \\ & & & & 1 & \\ & & & & -1 & 1 \end{bmatrix}$$

This matrix operates on a vector defined at regularly spaced, discrete points along the x axis, but whose z and t dependence may be either continuous or discrete. Given this definition of B, the restriction $0 \leq \beta \leq 1/4$ will guarantee the existence of the causal inverse operator derivative can be obtained by setting $\beta = 1/4$.

The new D_x can be used to build a Hermitian second derivative operator that provides a good approximation to the continuous second derivative over a wide range of wavenumbers. Forming the product $-D_x^H D_x$, simple algebra shows that

$$-D_x^H D_x f(x,z,t) = \frac{-1}{(\Delta x)^2} B^H B \left[I - \beta\, B^H B\right]^{-1} f(x,z,t)$$

where use has been made of the result that $B + B^H = B^H B = B B^H$. If the operator $T = BB^H$, is defined, then the matrix representation of T will have 2's on its diagonal and -1's on both its super- and sub-diagonals. Thus T, in matrix form, looks like

$$T = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

The second derivative operator can be approximated with a rational function of T. Substituting a T for every $B^H B$ in the expression for $-D_x^H D_x f(x,z,t)$, leads to the equality

$$-D_x^H D_x f(x,z,t) = \frac{-1}{(\Delta x)^2} T \left[I - \beta T\right]^{-1} f(x,z,t)$$

Finally, it may be desirable to carry out time differentiation in the frequency domain. This is because frequency is an eigenvalue of the wave equation operator, so the frequency components of a wave will propagate independently of one another. Use of this independence property decreases the I/O costs of a migration program, and increases the accuracy and bandwidth for which functions can be differentiated. If the Fourier transform of f(t) is defined by

$$F(\omega) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} f(t)$$

then the inverse of causal integration is equivalent to multiplication by $i\omega + \varepsilon$ in the limit $\varepsilon \to 0+$. To the extent that inversion of causal integration is causal differentiation, multiplication by $i\omega+\varepsilon$ is the frequency domain equivalent to causal differentiation in the time domain. Similarly, anti-causal differentiation is roughly equivalent to multiplication by the complex factor $i\omega-\varepsilon$.

One-way anti-causal wave equations

Migration of upwards traveling waves is a process that is anti-causal in time so $-D_t^H$ and not D_t will be used for temporal differentiation. With the notation and concepts introduced in the last section the two-way wave equation for propagating backwards in time can be written as

$$-D_z^H \frac{1}{\rho} D_z P - D_x^H \frac{1}{\rho} D_x P - \frac{1}{\kappa} * (-D_t)^{2H} P = 0$$

where the asterisk denotes a convolution with respect to time. It will be convenient to work in the frequency domain so that the time domain convolution does not complicate the algebra. To avoid the introduction of unnecessary new symbols, let P stand for both the pressure wave and its Fourier transform with respect to time. Define

$$\frac{1}{K(x,z,\omega)} = \int_0^\infty dt \; \frac{1}{\kappa}(x,z,t) \; e^{-i\omega t}$$

so that the wave equation can be written

$$-D_z^H \frac{1}{\rho} D_z P - D_x^H \frac{1}{\rho} D_x P - \frac{1}{K}(i\omega-\varepsilon)^2 P = 0$$

Since K may be frequency dependent, both phase velocity and acoustic slowness may be frequency dependent.

With a change of the dependent variable on which the wave equation operates, the parameters of the medium can be grouped together. This change will be advantageous when continued fractions are introduced into the discussion, because one consequence will be that the coefficients of the continued fraction will commute with one another. To make the change of variable, premultiply equation (1) by $K^{1/2}$ and introduce $P/K^{1/2}$. Since the extrapolation will be in the z direction the x and t derivatives are transposed to the right side of the equality

$$-K^{1/2}D_z^H\frac{1}{\rho}D_zK^{1/2}\frac{P}{K^{1/2}} = K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2}\frac{P}{K^{1/2}} + (i\omega-\varepsilon)^2\frac{P}{K^{1/2}} \qquad (2)$$

To get a one-way wave equation from equation (2), the left side of the equality must be first order in either D_z or D_z^H. For downward continuation, which is causal in depth, the left side of the equality must contain D_z only. Following Clayton (1981), make the approximation

$$K^{1/2}D_z\frac{1}{\rho}D_zK^{1/2}\frac{P}{K^{1/2}} \approx -K^{1/2}D_z^H\frac{1}{\rho}D_zK^{1/2}\frac{P}{K^{1/2}}$$

$$\approx K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2}\frac{P}{K^{1/2}} + (i\omega-\varepsilon)^2\frac{P}{K^{1/2}} \qquad (3)$$

This equation has the proper causality properties, but is still second order in z. It requires consistent, second order boundary conditions at $z = 0$ and at $z = \infty$.

One way to decrement the order of differential equation (3) is to take the square root of the operators on both sides of equation (3). Unfortunately, $K^{1/2}D_z\rho^{-1}D_zK^{1/2}$ does not have a square root with a simple analytic expression. An alternative approach is to consider arbitrary functions F and G of K and ρ and search for an approximation of the form FD_zGFD_zG. The approximation should equal the sum of $K^{1/2}D_z\rho^{-1}D_zK^{1/2}$ and an unknown multiplicative operator. The error involved in such an approximation will affect amplitudes only, the distortion becoming important when the z derivative of either K, ρ, or V becomes large. Solving for the unknown functions F and G, the only causal operator that will do the job is $V^{1/2}D_zVD_zV^{1/2}$. The error-inducing multiplicative operator that will be neglected, temporarily denoted by C, is given by the expression

$$C = K^{1/2} D_z^H \left(\frac{1}{\rho} D_z (K^{1/2}) \right) - V^{1/2} D_z (V D_z (V^{1/2}))$$

where the parentheses are meant to suggest the range of action of the derivatives in the expression. Replacing the operator on the left side of equation (3) with its causal approximation, and then dropping C yields a two-way wave equation that propagates waves in the positive z-direction only. Since migration pushes an upgoing wavefield in the positive z-direction, the use of causal z-derivatives is a step towards the formulation of a one-way wave equation. The two-way wave equation now under consideration is

$$\left[V^{1/2} D_z V^{1/2} \right]^2 \frac{P}{K^{1/2}} = K^{1/2} D_z^H \frac{1}{\rho} D_z K^{1/2} \frac{P}{K^{1/2}} + (i\omega - \varepsilon)^2 \frac{P}{K^{1/2}}$$

The conversion of this causal two-way wave equation into a causal one-way wave equation is achieved by taking the square roots of the operators on both sides of the equality. As is well known, any complex number has two square roots whose sum is equal to zero. Similarly, there are 2^n square roots of an n by n matrix operator. Fortunately, the physics of wave propagation will restrict the decision of which square root to use to those in

$$V^{1/2} D_z V^{1/2} \frac{P}{K^{1/2}} = \pm \left[K^{1/2} D_z^H \frac{1}{\rho} D_z K^{1/2} + (i\omega - \varepsilon)^2 \right]^{1/2} \frac{P}{K^{1/2}} \qquad (4)$$

up to an ambiguity in sign. Any other square root choice must partition the wavefield into its z-wavenumber components and propagate some of these components upwards, some downwards. One-way wave equations propagate all components either downwards or upwards. The sign ambiguity can be resolved by choosing the propagation direction. Suppose P and the medium parameters are such that the derivatives with respect to x can be ignored. When this is done, the operator on the right of the last equation is the square root of the $(i\omega - \varepsilon)^2$, that one might expect to equal $(i\omega - \varepsilon)$. Neglecting z derivatives of V and K, $D_z P \approx \pm(i\omega - \varepsilon)\Lambda P$. For an upgoing wave the '+' sign must be chosen over the '-'.

This is evident as long as we agree that the principal square root of a complex scalar z should be defined so that

Re z >= 0	Im z = 0	--->	Re z $^{1/2}$ >= 0	Im z $^{1/2}$ = 0
	Im z > 0	--->	Re z $^{1/2}$ > 0	Im z $^{1/2}$ >= 0
Re z < 0	Im z = 0	--->	Re z $^{1/2}$ = 0	Im z $^{1/2}$ >= 0
	Im z < 0	--->	Re z $^{1/2}$ < 0	Im z $^{1/2}$ > 0

These rules are enough to guarantee that the square root of $(i\omega-\varepsilon)^2$ will have a non-positive real part. Since the differential equation under consideration is to be integrated in the positive z-direction when migrating, the negative real part of the square root of $(i\omega-\varepsilon)^2$ is a necessary condition for the stability of the migration.

Returning to the design of a one-way wave equation for migrating upgoing waves, a formal differential equation that will do the job is obtained by choosing the '+' sign in equation (4). The result is

$$V^{1/2}D_z V^{1/2}\frac{P}{K^{1/2}} = \left\{(i\omega-\varepsilon)-(i\omega-\varepsilon)+\left[(i\omega-\varepsilon)^2+K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}\right]^{1/2}\right\}\frac{P}{K^{1/2}} \quad (5)$$

Through a change of state variable it is possible to get a differential equation in normal form that can be approximately solved with the help of the Crank-Nicolson approximation. The desired form is $D_z f = Op f$ and can be obtained in two steps by grouping $V^{1/2}$ with $P/K^{1/2}$ and then pre-multiplying both sides of the equation by $\Lambda^{1/2}$. The result is the partial differential equation

$$D_z\frac{V^{1/2}P}{K^{1/2}} = \Lambda^{1/2}\left\{(i\omega-\varepsilon) - (i\omega-\varepsilon) + \left[(i\omega-\varepsilon)^2 + K^{1/2}D_x^H\frac{1}{\rho}D_x K^{1/2}\right]^{1/2}\right\}\Lambda^{1/2}\frac{V^{1/2}P}{K^{1/2}}$$

The peculiar term $(i\omega-\varepsilon) - (i\omega-\varepsilon)$ in the braces on the right side of the equality is placed there because an accurate solution scheme does not solve the migration equation directly. Instead, the migration equation is split into two partial differential equations, a phase shift equation and a focusing equation. The two pieces are solved for alternately at each z-step. Up to second order terms in Δz, the sampling rate along the z axis, the results of solving the migration split and unsplit will be the same. For finite Δz, however, the split scheme enjoys clear superiority over the unsplit method of solution. This is because the Crank-Nicolson approximation for D_z is valid only for small z wavenumbers. Examination of the dispersion relation for the unsplit equation shows that its z wavenumbers are largest for vertically traveling waves. The Crank-Nicolson approximation behaves poorly for the wavenumbers we want it to be at its best behavior for. Splitting remedies this by calling for the alternate solution of the two differential equations

$$D_z \frac{V^{1/2}P}{K^{1/2}} = (i\omega-\varepsilon)\Lambda \frac{V^{1/2}P}{K^{1/2}} \tag{6a}$$

$$D_z \frac{V^{1/2}P}{K^{1/2}} = \Lambda^{1/2}\left\{ -(i\omega-\varepsilon) + \left[(i\omega-\varepsilon)^2 + K^{1/2}D_x^H \frac{1}{\rho} D_x K^{1/2}\right]^{1/2}\right\}\Lambda^{1/2}\frac{V^{1/2}P}{K^{1/2}} \tag{6b}$$

The first equation of this split shifts the wavefield vertically through $\omega\Lambda z / 2\pi$ samples. The analytic solution for the wavefield at depth z given the wavefield at depth $z-\Delta z$ is $V^{1/2}P/K^{1/2}(x,z,\omega) = \exp[(i\omega-\varepsilon)\Lambda\Delta z]\ V^{1/2}P/K^{1/2}(x,z-\Delta z,\omega)$. Equation (6b) of the split pair is a focusing equation that governs the diffraction effects associated with extrapolation of the state variable $V^{1/2}P/K^{1/2}$ from a depth $z-\Delta z$ to a depth z. The phase shift equation does most of the work when the dip angle is small. The focusing equation becomes important when the dip angle is either large or changing as a function of the lateral coordinate x.

Ma's method

Equation (6b) can be cast into discrete form by applying the Crank-Nicolson approximation and then expanding the square root as a continued fraction. This procedure leads to algebraic difficulties for equations of higher order than the 45-degree equation when V is allowed to vary as a function of x. An alternative method, first suggested by Ma (1982), is to expand the square root in equation (6b) in its continued fraction before discretizing the z axis. This continued fraction can, of course, be approximated by one of its approximants. The approximant can then be expanded by the method of partial fractions and the expansion split into its component pieces. The component pieces can be alternately solved at each z-step taken during a migration.

With this program in mind, it is appropriate to focus attention on the quantity in braces in equation (6b). The operator in question

$$-(i\omega-\varepsilon)+\left[(i\omega-\varepsilon)^2+K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2}\right]^{1/2} \tag{7}$$

will be approximated by an approximant of its continued fraction expansion. It will be assumed that $(i\omega-\varepsilon)$ is a constant diagonal matrix so that the eigenvectors of $K^{1/2}D_x^H\rho^{-1}D_xK^{1/2}$ will also be eigenvectors of $(i\omega-\varepsilon)$. The operator $K^{1/2}D_x^H\rho^{-1}D_xK^{1/2}$ is Hermitian when K is real, the usual case in applications. Hermitian operators are complete, have real eigenvalues, and eigenvectors of distinct eigenvalues are orthogonal. Finally, independent eigenvectors that share an eigenvalue can always be built so that they too are orthogonal. From the discussion of the last chapter, the eigenvalues of the sum of $(i\omega-\varepsilon)^2$ with this Hermitian operator will have a negative imaginary part. Consequently, the square root will have a negative real part and the migration operator will be stable. The continued fraction for the operator in equation (7)

$$\cfrac{I}{2(i\omega-\varepsilon)+\cfrac{I}{2(i\omega-\varepsilon)+\cdots}K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2}}K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2} \tag{8}$$

The approximants of continued fraction (8) will be needed for the development of 45-degree and higher order migration operators. By generalizing the argument to be developed here, it can be shown that migration's computational costs increases at every other increase in the order of the approximant of (8) employed. Therefore, every other approximant of the operator in continued fraction (8) will be considered. Denoting the approximants by R_k, these can be generated using the recurrence

$$R_1 = \cfrac{I}{2(i\omega-\varepsilon)+\cfrac{I}{2(i\omega-\varepsilon)\psi}K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2}}K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2} \tag{9a}$$

$$R_{k+1} = \cfrac{I}{2(i\omega-\varepsilon)+\cfrac{I}{2(i\omega-\varepsilon)+R_k}K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2}}K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2} \tag{9b}$$

where ψ is chosen to improve the approximation to the dispersion relation and to introduce dip filtering. The sequence of approximants generated by equations (9) can be obtained by truncating the continued fraction in (8). Equation (9a) of the recursion can be used to build a 45-degree equation, while equation (9b) yields equations of ever increasing order. Each element of the recursion is a fraction whose denominator can be cleared of fraction bars by using the fundamental recurrence relation for continued fractions. The fraction, once its denominator has been cleared, can be expanded by the method of partial fractions. As it stands, however, the coefficients will be frequency dependent. To get a partial fraction expansion that is independent of ω, let S represent the operator $(i\omega-\varepsilon)^{-2}K^{1/2}D_x^H\frac{1}{\rho}D_xK^{1/2}$. The recursion relations can be rewritten, using S, as

$$R_1 = (i\omega-\varepsilon)S\,\cfrac{I}{2I+\cfrac{I}{2\psi}S} \tag{10a}$$

$$R_{k+1} = (i\omega-\varepsilon)S \frac{I}{2I+\frac{I}{2I+(i\omega-\varepsilon)^{-1}R_k}S} \tag{10b}$$

Each element of this sequence of approximants is the product of $(i\omega-\varepsilon)S$ with a rational function of S. The order of the numerator is always one less than that of the denominator. Barring multiple roots, each R_k can be expanded as a sum of fractions of the form

$$R_k = \sum_{j=1}^{k} \frac{a_j^{(k)}(i\omega-\varepsilon)S}{b_j^{(k)}+S}$$

each term of which is a 45-degree operator. Plugging the partial fraction expansion into equation (6b) yields another focusing equation

$$D_z \frac{V^{1/2}P}{K^{1/2}} = \Lambda^{1/2}\left\{\sum_{j=1}^{k} \frac{a_j^{(k)}(i\omega-\varepsilon)S}{b_j^{(k)}+S}\right\}\Lambda^{1/2}\frac{V^{1/2}P}{K^{1/2}} \tag{11}$$

The solution of this differential equation is obtained by splitting it into its component terms, solving each split equation at every z-step. Once again, this strategy is accurate to second order terms in Δz, the discretization parameter for the depth axis.

The discretization of the focusing equation

Each equation of the set of differential equations generated by splitting equation (11) is a 45-degree equation. The discrete analogue of this equation has yet to be derived. With the assumption of constant density, the operators that need to be discretized are the spatial derivatives. In particular, discrete representations of the first partial derivative with respect to z and the second partial derivative with respect to x will be required.

The discrete representation of the first derivative with respect to z that is usually employed is the causal Crank-Nicolson approximation. The error introduced by using this approximation is third order in Δz for small wavelengths. The bandwidth over which the Crank-Nicolson approximation is valid is narrow, but the cheapest remedy for this is to decrease the magnitude of Δz.

Experience shows that a second derivative operator that is valid over as wide a bandwidth as possible is a important component in building a migration scheme - more important than fitting the dispersion relation for upgoing waves to high order. Borrowing from the section on derivatives and causality, a discrete second difference operator with a wide bandwidth is found by implementing

$$-D_x^H D_x f(x,z,t) = \frac{-1}{(\Delta x)^2} T(I - \beta T)^{-1} f(x,z,t)$$

The denominator of the operator on the right of the equality is a bit of a nuisance, but it is a removable nuisance. Good choices for β will be discussed elsewhere.

Once a set of derivative operators has been selected, it is necessary to introduce them into an equation of the split from Ma's focusing equation. It will also be convenient to introduce a new state variable, say Q, to stand in the place of the $V^{1/2}P/K^{1/2}$s. Once again, an S will denote the operator $(\Delta x)^{-2} VT(I-\beta T)^{-1} V/(i\omega-\varepsilon)^2$. Referring back to equation (11), the representative differential equation takes the form

$$\frac{2}{\Delta z}(Q(z)-Q(z-\Delta z)) = \Lambda^{1/2}\frac{a_j^{(k)}(i\omega-\varepsilon)S}{b_j^{(k)}+S}\Lambda^{1/2}(Q(z)+Q(z-\Delta z))$$

where the ω and x dependence of Q has been suppressed. At this point Q is a discrete function of the three coordinates, z, ω, and x. The unknown in the discretized form of equation (11) is the field Q at depth z. Bringing the terms involving $Q(z)$ to the left of the equality and terms that are dependent on $Q(z-\Delta z)$ to the right, we obtain

$$\left[\frac{2}{\Delta z} - \Lambda^{1/2}\frac{a_j^{(k)}(i\omega-\varepsilon)S}{b_j^{(k)}+S}\Lambda^{1/2}\right] Q(z) = \left[\frac{2}{\Delta z} + \Lambda^{1/2}\frac{a_j^{(k)}(i\omega-\varepsilon)S}{b_j^{(k)}+S}\Lambda^{1/2}\right] Q(z-\Delta z)$$

This equation cannot be considered easily solvable yet, since it requires the inversion of the matrix $b_j^{(k)}+S$. Fortunately, the denominators of the operators on both sides of the equality can be cleared by premultiplying by $b_j^{(k)}+SV^{1/2}$. The result of this operation is the difference equation

$$\left[\frac{2}{\Delta z}(b_j^{(k)}+S)\,V^{1/2} - a_j^{(k)}(i\omega-\varepsilon)S\Lambda^{1/2}\right] Q(z)$$

$$= \left[\frac{2}{\Delta z}(b_j^{(k)}+S)\,V^{1/2} + a_j^{(k)}(i\omega-\varepsilon)S\Lambda^{1/2}\right] Q(z-\Delta z)$$

Unfortunately, this equation is not easily solvable either. The problem this time is that the operator $S = (\Delta x)^{-2} VT(I-\beta T)^{-1} V/(i\omega-\varepsilon)^2$ still has a matrix inverse in it. Once again, the denominator can be cleared of matrices by premultiplication, this time by $(I-\beta T)\Lambda$. Premultiplication of S by this factor yields the simple operator $(i\omega-\varepsilon)^{-2}(\Delta x)^{-2}TV$. Premultiplication of both sides of the difference equation under consideration yields

$$\left[b_j^{(k)}(I-\beta T)+(i\omega-\varepsilon)^{-2}(\Delta x)^{-2}TV^2-\frac{\Delta z a_j^{(k)}}{2(i\omega-\varepsilon)(\Delta x)^2}TV\right]\Lambda^{1/2}Q(z)$$

$$=\left[b_j^{(k)}(I-\beta T)+(i\omega-\varepsilon)^{-2}(\Delta x)^{-2}TV^2+\frac{\Delta z a_j^{(k)}}{2(i\omega-\varepsilon)(\Delta x)^2}TV\right]\Lambda^{1/2}Q(z-\Delta z) \quad (12)$$

where the equation has been scaled to make the quantity in brackets dimensionless. The solution of this linear equation is slightly cheaper if yet another intermediate variable Q', equal to $\Lambda^{1/2}Q$, is introduced. Equation (12) becomes a linear equation in the unknown Q', from which Q is easily determined.

If the operators on both sides of the equality were made dimensionless, then the downward continuation process would be completely insensitive to choices of measurement units. This is certainly a desirable feature, but in the presence of x-variable velocity, it can be achieved only at the cost of more arithmetic. In particular, a dimensionless version of equation (12) can be obtained by premultiplying both sides of the equality by $V^{1/2}$.

Reassembling the pieces for a z-step

In the preceding sections the one-way wave equation was split into simpler differential equations. The number of differential equations is determined by the order of approximation to the dispersion relation of an upgoing wavefield. Denote the number of equations in the split by $K+1$, where K is a positive integer. Given the wavefield $Q(z-\Delta z)$ at depth $z-\Delta z$, we want to find $Q(z)$. This is done by following the procedure found in table 4.1.

1. $Q_{-1} = \exp\,[(i\omega-\varepsilon)\Lambda\Delta z\,]\; Q(z-\Delta z)$

2. $Q_0 = \Lambda^{1/2} Q_{-1}$

3. $r_l = \dfrac{2\,Q_0(2)\,Q_0^*(3)}{|\,Q_0(2)\,|^2 + |\,Q_0(3)\,|^2} \qquad r_r = \dfrac{2\,Q_0(n_x-1)\,Q_0^*(n_x-2)}{|\,Q_0(n_x-1)\,|^2 + |\,Q_0(n_x-2)\,|^2}$

4. For $j = 1$ to K

$$\left[b_j^{(K)}(I-\beta T) + (i\omega-\varepsilon)^{-2}\, TV^2 - \frac{\Delta z a_j^{(K)}}{2(i\omega-\varepsilon)(\Delta x)^2}\, TV \right] Q_j =$$

$$\left[b_j^{(K)}(I-\beta T) + (i\omega-\varepsilon)^{-2}\, TV^2 + \frac{\Delta z a_j^{(K)}}{2(i\omega-\varepsilon)(\Delta x)^2}\, TV \right] Q_{j-1}$$

$Q_j(1) = r_l\, Q_j(2) \qquad Q_j(n_x) = r_r\, Q_j(n_x-1)$

5. $Q(z) = V^{1/2} Q_K$

TABLE 4.1. A finite difference z-step schematic. A five step procedure for downward continuing a wavefield through a single z step of size Δz. $Q(z-\Delta z)$ is the input wavefield at depth $z-\Delta z$. Similarly, $Q(z)$ is the output wavefield at depth z. The subscripted Q's are auxiliary vectors used in implementing the finite difference scheme. When present, the argument of a subscripted Q is an index of position along the x axis. For instance, $Q_0(n_x-1)$ is the (n_x-1)-th component of the vector Q_0.

Step 1 in table 4.1 is a vertical phase shift of the input data. It is the solution of equation (6a), the result of splitting the full one-way wave equation into shifting and focusing equations. Step 2 simply scales the wavefield by the square root of the acoustic slowness. Side boundary coefficients are computed in the next step. Step 4 calls for the solution of the K equations generated by applying Ma's method to the focusing equation. Suitable boundary conditions are applied at both x boundaries. Finally, the result is scaled by the square root of the acoustic velocity.

Side boundaries

Side boundary conditions have been neglected so far. Ma's method allows an extension of the boundary conditions used in 45-degree equation codes to higher order migration algorithms. Zero-value and zero-slope boundaries are trivial, so all that will be considered here are absorbing side boundaries that adapt to the data.

Steps 3 and 4 in table 4.1 carry out different pieces of an adaptive scheme for absorbing side boundaries. The boundary is included in the discretization mesh. Denoting the number of points along the x axis by n_x, the boundaries occur at the first and n_x-th point along that axis. Values of the wavefield at the

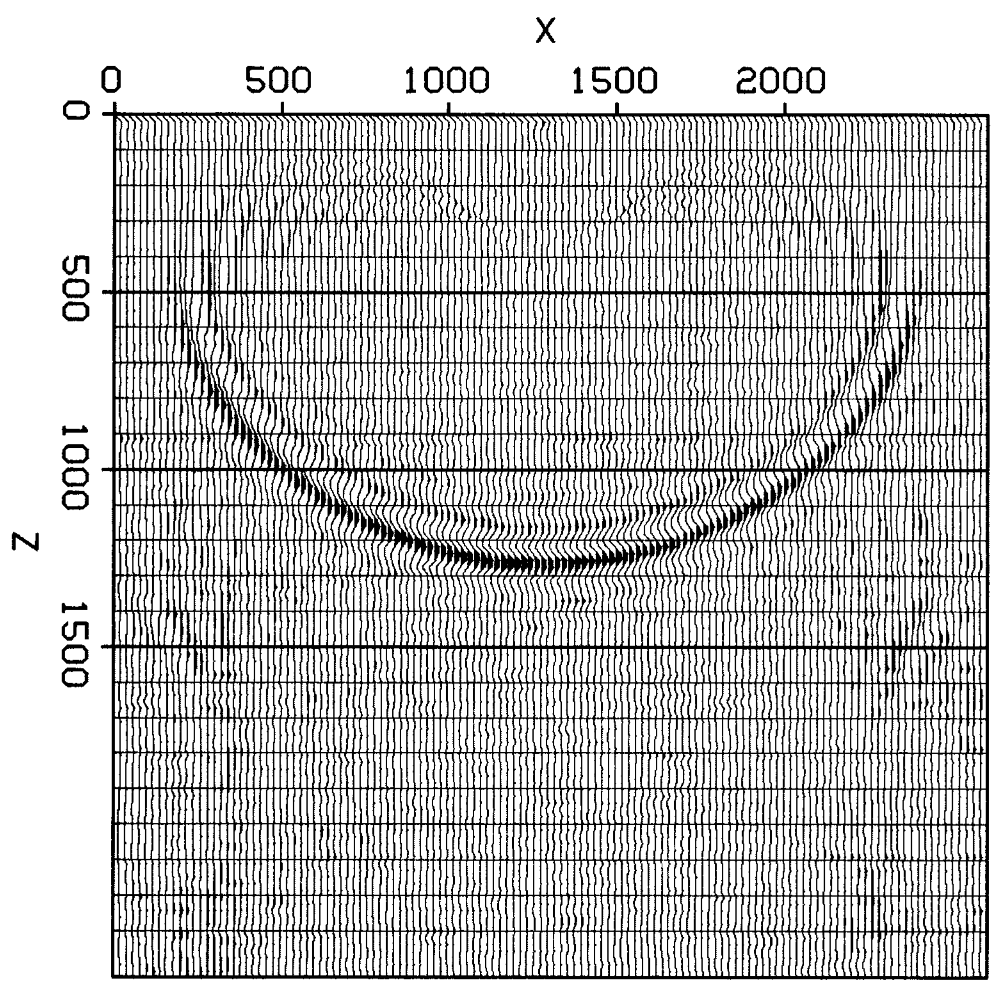

FIGURE 4.1. Impulse response of the 45 migration operator. With ψ = 0.73 and a frequency variable β, the first of Ma's migration operators is applied to a band limited impulse. The input was generated by convolving a spike with a three-point Nyquist suppressing filter in both the time and space directions.

x indices 2, 3, n_x-2, and n_x-1 are used as inputs to the Burg algorithm. The reflection coefficients r_l and r_r minimize the sum of local forward and backward prediction errors.

An explanation for the boundary condition coefficients can be found by considering the boundary at the right hand side of the grid, at x point number n_x. Consider the differential equation

$$\left[D_x - \sin(\theta)\, \Lambda D_t\right] Q = 0$$

for continuous Q. This equation has solutions of the form $Q = Q(x \sin\theta + Vt)$, the wavefronts of which move in the direction the positive x direction as t evolves in the negative temporal direction. Since migration is anti-causal in time, this is the right type of evolution equation for waves that are incident on the right hand side boundary. With the Fourier transform convention used in this chapter the solutions of the side boundary differential equation, when put into the frequency domain, are proportional to $\exp[(i\omega-\varepsilon)\Lambda x \sin\theta]$. Considering the discrete Q_j again, we have

$$Q_j(n_x) = \exp[(i\omega-\varepsilon)\Lambda\Delta x \sin\theta]\; Q_j(n_x-1) \tag{13}$$

$$Q_j(n_x) = r_r\, Q_j(n_x-1) \tag{14}$$

Thus, the reflection coefficient r_r can be identified as an estimate of the complex exponential $\exp[(i\omega-\varepsilon)\Lambda\Delta x \sin\theta]$. In the absence of aliasing in the x direction, the complex exponential has a positive imaginary part. The positive imaginary property can be used as a check on the quality of the reflection coefficient estimate r_r. If $\mathrm{Im}\, r_r$, computed per table 4.1, is positive then equation (13) is appropriate. Otherwise, the reflection coefficient is judged to be poor (corresponding to an aliased wave or to a wave propagating in the wrong direction), and equation (14) is used.

Finally, it should be noted that zero-value and zero-slope boundary conditions can be had by setting r_r equal to 0 and 1, respectively.

A stability analysis for one-way wave equation operators

The one-way wave equation approximations introduced here were based on a continued fraction. Other approximations could have been used, a Taylor series for example. Continued fraction approximants to the wave equation have an

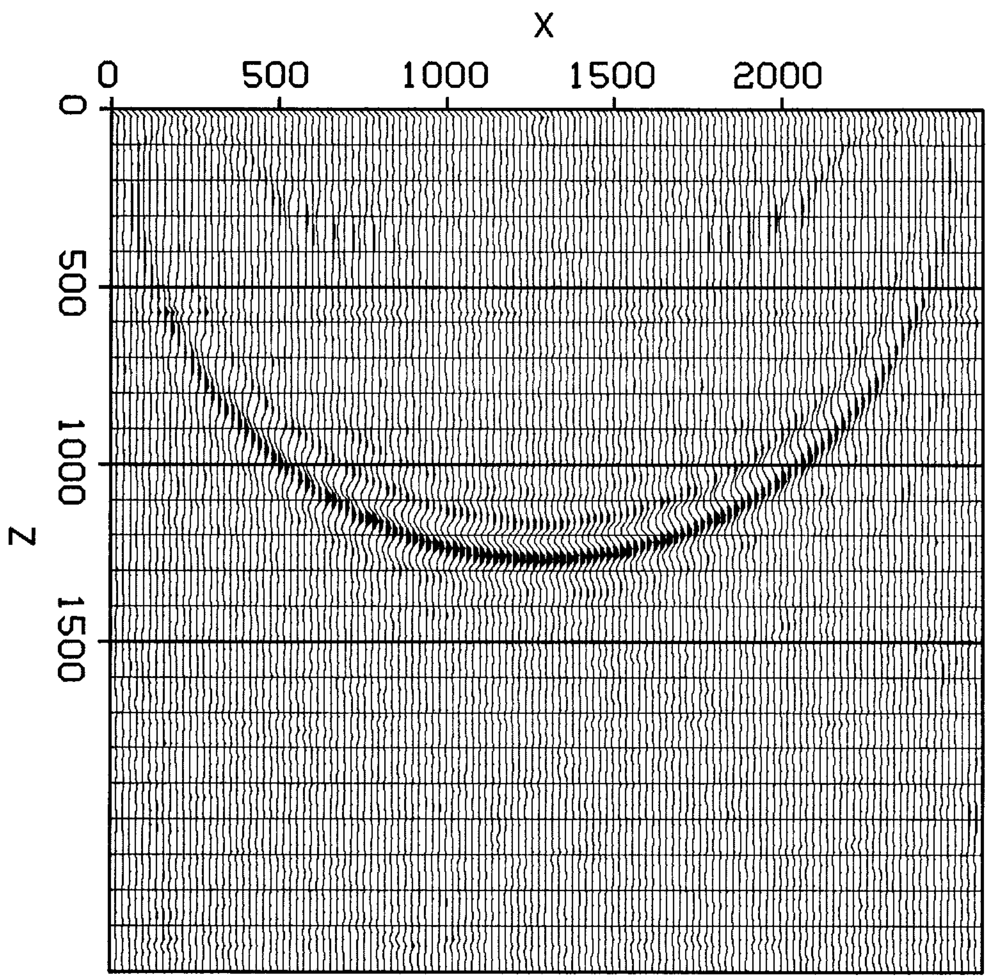

FIGURE 4.2. Impulse response for Ma's second operator. With ψ = 0.6553 and a frequency variable β, the second of Ma's migration operators is applied to a band limited impulse. The operator is of order one higher than that of the 45 equation. The input here is the same as that for figure 4.1.

advantage over their competitors, however, in that their stability can be analyzed. In this section, stability is considered for a wave equation operator with reflecting side boundaries.

A good starting point for the analysis is a version of equation (6b) which has been discretized along the x axis. The operator on the right-hand side of this equation is of the form

$$R_\infty = \Lambda^{1/2}\left\{-(i\omega-\varepsilon)+\left[(i\omega-\varepsilon)^2+VT(I-\beta T)^{-1}V\right]^{1/2}\right\}\Lambda^{1/2}$$

where ε is a small positive and real number. The approximants of this operator's continued fraction expansion can be built up by using the recurrence

$$R_1 = \Lambda^{1/2}\frac{VT(I-\beta T)^{-1}V}{2(i\omega-\varepsilon)\psi}\Lambda^{1/2}$$

$$R_{k+1} = \Lambda^{1/2}\frac{VT(I-\beta T)^{-1}V}{2(i\omega-\varepsilon) + V^{1/2}R_k V^{1/2}}\Lambda^{1/2}$$

where ψ is a complex constant with positive real and positive imaginary parts.

The discussion of stability will begin by demonstrating that the Crank-Nicolson approximation to (6b) is stable when the eigenvalues of R_k have non-positive real parts. The stability proof will finish with a proof that the eigenvalues of R_k do have non-positive real parts.

Equation (6b) is an equation of the form $D_z Q = R_k Q$. Use of the Crank-Nicolson approximation to the z derivative, leads to a difference equation that can be written in the form

$$Q(z) = \left[I - \frac{\Delta z}{2} R_k \right]^{-1} \left[I + \frac{\Delta z}{2} R_k \right] Q(z-\Delta z) \tag{15}$$

where $Q(z-\Delta z)$ is a known quantity and $Q(z)$ is an unknown. The matrices in this equation commute with one another, so they share a common set of eigenvectors. If one eigenvector is selected and its corresponding eigenvalue for the matrix R_k is denoted by λ_0, the eigenvalue for the matrix on the right-hand side of the equation (15) is equal to $[2-\lambda_0\Delta z]/[2+\lambda_0\Delta z]$. The modulus of this quantity is less than unity, so the eigenvectors must decay asymptotically to zero under the influence of equation (15). Since this result holds for an arbitrary eigenvector of the matrix on the right-hand side of equation (15), downward continuation with equation (15) must be an asymptotically stable process.

To start the demonstration that the eigenvalues of R_k have non-positive real parts, consider an arbitrary complex vector x and the sum $x^H(R_1 + R_1^H)x$. The sum can be expanded as

$$x^H \Lambda^{1/2} \left[\frac{VT(I-\beta T)^{-1} V}{2(i\omega-\varepsilon)\psi} + \frac{V(I-\beta T)^{-1} TV}{2(i\omega+\varepsilon)\psi^H} \right] \Lambda^{1/2} x$$

since $\Lambda^{1/2}$, V, and T are Hermitian symmetric matrices. The commutativity of T and $I-\beta T$ implies that the sum is also equal to

$$x^H \Lambda^{1/2} VT(I-\beta T)^{-1} V \Lambda^{1/2} x \ \mathrm{Re}\left[\frac{1}{(i\omega-\varepsilon)} \psi \right]$$

where $x^H \Lambda^{1/2} VT(I-\beta T)^{-1} V \Lambda^{1/2} x$ is a non-negative definite quadratic form. The complex number $1/((i\omega-\varepsilon)\psi)$ has a positive real part by construction, so the sum $x^H(R_1 + R_1^H)x$ is a real number less than or equal to zero. If x is chosen so that it is an eigenvector of R_1 with a corresponding eigenvalue λ_x, then

$$\begin{aligned} x^H\left[R_1^H + R_1\right]x &= x^H R_1^H x + x^H R_1 x \\ &= x^H \lambda_x^H x + x^H \lambda_x x = 2x^H x \mathrm{Re}\left[\lambda_x\right] \end{aligned}$$

Since $x^H x$ is a positive real number and its product with the real part of λ_x is

$$V^{1/2}R_{k+1}V^{1/2} = \frac{VT(I-\beta T)^{-1}V}{2(i\omega-\varepsilon)+V^{1/2}R_k V^{1/2}}$$

where $V^{1/2}R_k V^{1/2}$, $(i\omega-\varepsilon)$, and $VT(I-\beta T)^{-1}V$ are normal and commute with one another. It follows that these three matrices and $V^{1/2}R_{k+1}V^{1/2}$ share a common set of eigenvectors. Choosing one such eigenvector, let λ_1 and λ_2 denote the corresponding eigenvalues for the matrices $V^{1/2}R_k V^{1/2}$ and $VT(I-\beta T)^{-1}V$, respectively. Since $VT(I-\beta T)^{-1}V$ is positive definite symmetric, λ_2 is real and positive. By the inductive hypothesis λ_1 has a non-positive real part. The corresponding eigenvalue for $V^{1/2}R_k V^{1/2}$ is equal to $\lambda_2/(2(i\omega-\varepsilon)+\lambda_1)$, a quantity with a non-positive real part. By mathematical induction $V^{1/2}R_k V^{1/2}$ is normal and has eigenvalues that have non-positive real parts for all positive integers k.

Because $V^{1/2}R_k V^{1/2}$ is normal and non-positive-real, R_k has eigenvalues with non-positive real parts. For an arbitrary complex vector x,

$$x^H\left[R_k^H+R_k\right]x = (\Lambda^{1/2}x)^H V^{1/2}R_k^H V^{1/2}(\Lambda^{1/2}x)+(\Lambda^{1/2}x)V^{1/2}R_k V^{1/2}(\Lambda^{1/2}x)$$

Since $V^{1/2}R_k V^{1/2}$ is normal and has non-positive real eigenvalues, its eigenvectors are orthogonal to one another. By expanding $\Lambda^{1/2}x$ in this orthogonal set of eigenvectors, one can easily show that the terms in the above sum have non-positive real parts and that their imaginary parts cancel one another. Thus, $x^H\left[R_k^H+R_k\right]x$ is non-positive. As before, this is enough to guarantee that the eigenvalues of R_k has a non-positive real part.