# CHAPTER 3

## The Algebra of Continued Fractions

The migration operators built in the next chapter will use a continued fraction with matrix operators for coefficients. A continued fraction of this type generates a sequence of rational forms called approximants. Thus, the algebra of these fractions and their approximants needs to be developed before proceeding with a discussion of migration operators. This chapter discusses an algorithm for generating sequences of approximants of continued fractions. Enough terminology is introduced to make the next chapter intelligible. The type continued fraction that is used in the design of migration operators is one whose matrix coefficients are complete and commute with one another. An algebra for studying this type continued fraction is developed in the final two sections.

### Continued fractions with matrix coefficients

The continued fractions usually considered by mathematicians have real or complex coefficients. For instance, such fractions are employed as Padé approximations to a function. Because of the attention paid them, the algebraic and analytic properties of continued fractions with scalar coefficients are well understood. The development of finite difference migration algorithms requires the extension of these properties to continued fractions with square matrix coefficients. Some of the simplest algebraic properties of this type continued fraction are developed in this section. Analytic properties, such as convergence, will be neglected.

The building blocks for the continued fractions considered in this thesis are square matrices. There are three sorts of coefficients, so it is appropriate to consider three collections of $N$ by $N$ matrix coefficients,

$$\left\{a_j\right\}_{j=1}^{\infty}, \quad \left\{b_j\right\}_{j=0}^{\infty}, \quad \left\{c_j\right\}_{j=1}^{\infty}$$

from which a continued fraction $F$ can be generated. To build $F$, consistently place $a$'s to the left of fraction bars, $b$'s to the left of addition signs, and $c$'s to the right of fraction bars. The result is

$$F = b_0 + a_1 \frac{I}{b_1 + a_2 \dfrac{I}{b_2 + a_3 \dfrac{I}{b_3 + \cdots} c_3} c_2} c_1 \tag{1}$$

where the $\cdots$ indicates that the structure repeats itself indefinitely. A matrix fraction with an identity matrix numerator is understood to represent the matrix inverse of its denominator. The existence of the matrix inverses involved will not be explored in this thesis.

An interpretation of the continued fraction in equation (1) can be made by considering $F$ to be the limit of the functional composition of an infinite sequence of linear fractional transformations $t_p, p = 0,1,2,3,\cdots$. These non-linear transformations are defined for $w$ belonging to a subset of the set of $N$ by $N$ complex matrices by the expressions

$$t_0(w) = b_0 + w$$

$$t_p(w) = a_p \frac{I}{b_p + w} c_p \qquad p = 1,2,3,\cdots$$

The argument $w$ must be restricted to the class of matrices for which the matrix inverse of $b_p + w$ exists. The existence of these inverses, once again, will be assumed. Linear fractional transformations can be combined via the operation of functional composition. For example, the resultant of operating on $t_1$ with $t_0$ is a transformation

$$t_0(t_1(w)) = b_0 + a_1 \frac{I}{b_1 + w} c_1$$

Similarly, the continued fraction $F$ can be considered to be the limit of a sequence of transformational compositions of linear fractional compositions evaluated at some value of $w$. If the limit is independent of this choice of $w$, then the continued fraction can be considered well-defined.

$$F = F(w) = \lim_{p \to \infty} t_0 t_1 t_2 \cdots t_p(w)$$

Convergence is often difficult to prove, even in the case where the continued fraction coefficients are real numbers. However, the one-way wave equation operator has a continued fraction with periodic, commuting $N$ by $N$ matrix coefficients, the analysis of which is easier than that of general continued fractions. In this special case, the convergence of the $N$ eigenmodes can be considered separately. The result of the analysis is that convergence is attained for propagating waves, but head wave modes diverge by oscillation. The convergence proof is tedious and not important enough to be included here, but the next chapter and the texts listed in the bibliography can be consulted for this purpose (Wall, 1972, Jones and Thron, 1980).

A continued fraction is a limit of a sequence of fractions. Most terms in the sequence have denominators with fractions in them. It turns out that there is an algorithm for computing the terms of the sequence recursively. Just as important for our purposes, the algorithm churns out the terms with denominators cleared of fraction bars. Denote the $k$-th cleared denominator by $B_k$, the corresponding numerator by $A_k$, and the $k$-th term of the sequence by $B_k^{-1} A_k$. The recurrence relates successive cleared denominators, numerators, and continued fraction coefficients according to the prescription:

$$F_k(w) = t_0 t_1 t_2 \cdots t_k(w) = \left[ B_k + w a_k^{-1} B_{k-1} \right]^{-1} \left[ A_k + w a_k^{-1} A_{k-1} \right]$$

$$A_{-1} = I \qquad A_0 = b_0 \qquad a_0 = I$$

$$B_{-1} = 0 \qquad B_0 = I$$

$$A_{k+1} = c_{k+1} a_k^{-1} A_{k-1} + b_{k+1} a_{k+1}^{-1} A_k \qquad k = 1,2,3,\ldots$$

$$B_{k+1} = c_{k+1} a_k^{-1} B_{k-1} + b_{k+1} a_{k+1}^{-1} B_k \qquad k = 1,2,3,\ldots$$

Now for some nomenclature: $A_n$ is the $n$-th numerator, $B_n$ is the $n$-th denominator, the ratio $B_n^{-1} A_n$ is the $n$-th approximant, $a_n$ is the $n$-th left partial numerator, $c_n$ is the $n$-th right partial numerator, and $b_n$ is the $n$-th partial denominator. The difference from the usual continued fraction theory lies in the distinction between left and right partial numerators.

**The fundamental recurrence for continued fractions**

The previous section showed that a continued fraction can be generated by a sequence of rational transformations $t_p$, where $p$ is a non-negative integer and the transformations are of the form $t_0(w) = b_0 + w$, $t_p(w) = a_p(b_p + w)^{-1} c_p$. Also, a recurrence relation was given for the functional compositions $t_0 t_1 t_2 \cdots t_k(w)$. In this section an inductive proof for this recurrence relation is presented.

The first term of the recurrence is $F_0(w)$. The partial left numerator $a_{-1}$, the first two numerators, $A_{-1}$ and $A_0$, and the first two denominators, $B_{-1}$ and $B_0$, were chosen so that

$$F_0(w) = \left[B_0 + w a_0^{-1} B_{-1}\right]^{-1} \left[A_0 + w a_0^{-1} A_{-1}\right] = \left[b_0 + wI\right]$$

The induction proceeds by assuming a structure for the $k$-th term in the sequence defining $F$. From this inductive hypothesis, it is shown that the $k+1$-st term has the same structure. By the principle of mathematical induction, the structure will hold for all $k$. The inductive hypothesis is that $F_k$ takes the form

$$F_k(w) = \left(B_k + w a_k^{-1} B_{k-1}\right)^{-1} \left(A_k + w a_k^{-1} A_{k-1}\right)$$

for an arbitrary $N$ by $N$ matrix input $w$. Identifying the various terms in the rational form yields a recurrence relation for the $A_k$'s and $B_k$'s. The next fraction,

$F_{k+1}(w)$, is defined by

$$F_{k+1}(w) \;=\; F_k t_{k+1}(w) \;=\; F_k(a_{k+1} \frac{I}{b_{k+1} + w} c_{k+1})$$

With a little algebra, paying close attention to the lack of commutativity among the various matrices, this expression can be simplified to look like

$$F_{k+1}(w) \;=\; \left[(c_{k+1}a_k^{-1}B_{k-1}+b_{k+1}a_{k+1}^{-1}B_k)+wa_{k+1}^{-1}B_k\right]^{-1}$$
$$\left[(c_{k+1}a_k^{-1}A_{k-1}+b_{k+1}a_{k+1}^{-1}A_k)+wa_{k+1}^{-1}A_k\right]$$

$$F_{k+1}(w) \;=\; \left(B_{k+1} + wa_{k+1}^{-1}B_k\right)^{-1}\left(A_{k+1} + wa_{k+1}^{-1}A_k\right)$$

Equating coefficients yields a recurrence for both the $A_k$'s and $B_k$'s involving the partial numerators and denominators:

$$A_{k+1} \;=\; c_{k+1}a_k^{-1}A_{k-1} + b_{k+1}a_{k+1}^{-1}A_k$$
$$B_{k+1} \;=\; c_{k+1}a_k^{-1}B_{k-1} + b_{k+1}a_{k+1}^{-1}B_k$$

The necessary initializations for this recurrence need to be found. To get the starting points, consider the cases in which $k = 0$ and $k = 1$.

$$t_0(w) \;=\; b_0 + w \;=\; \left(I + 0w\right)^{-1}\left(b_0 + Iw\right)$$

$$t_1(w) \;=\; (b_1a_1^{-1}+wa_1^{-1})^{-1}(b_1a_1^{-1}b_0+c_1+wa_1^{-1}b_0)$$

Equating coefficients again and assuming non-zero $a_0$,

$$A_{-1} \;=\; a_0 \qquad A_0 \;=\; b_0$$
$$B_{-1} \;=\; I \qquad B_0 \;=\; I$$

will provide a suitable recurrence initialization. For convenience, set $a_0$ and therefore $A_{-1}$ equal to identity operators.

From the fundamental recurrence it can be seen that pre-multiplying $b_{k+1}$ and $c_{k+1}$ by the same non-singular matrix will not change the approximants of the continued fraction. The same can be said for post-multiplication of $c_{k+1}$ and $a_k$ and for post-multiplication of $b_{k+1}$ and $a_{k+1}$.

**Algebras of diagonalizable matrices with a common set of eigenvectors**

The analysis of matrices is harder than that of scalars because matrix multiplication is not necessarily commutative, although some subsets of the set of all matrices form an algebraic structure in which matrix multiplication is commutative. One such set is formed by a subset of all matrices that share a common and complete set of eigenvectors. Elements from such a set will be useful in the design of downward continuation operators.

Suppose $M$ and $N$ have a common set of eigenvectors. Then $M$ and $N$ can be expanded in terms of a common non-singular matrix $T$ and diagonal matrices of eigenvalues $\Lambda_M$ and $\Lambda_N$, respectively. Define **S** to be the set of all such matrices that can be so expanded. The following rules apply:

(1) $M + N$ belongs to **S** ,

(2) $M\ N$ belongs to **S** ,

(3) $MN = NM$,

(4) if $z$ is a complex scalar then $z\ M$ belongs to **S** , and

(5) if $n$ is a non-negative integer then $M^n$ belongs to **S** ,

subject to the existence of $M^{-1}$ where appropriate. These relations follow from the following considerations

$$M + N = T\ \Lambda_M\ T_M^{-1} + T\ \Lambda_N\ T_M^{-1} = T\ (\Lambda_M + \Lambda_N)\ T_M^{-1}$$

$$M\ N = T\ \Lambda_M\ T_M^{-1}\ T\ \Lambda_N\ T_M^{-1} = T\ (\Lambda_M\ \Lambda_N)\ T_M^{-1} = T\ (\Lambda_N\ \Lambda_M)\ T_M^{-1} = N\ M$$

$$z\ M = z\ T\ \Lambda_M\ T_M^{-1} = T\ (z\,\Lambda_M)\ T_M^{-1}$$

$$M^{-1} = (T\ \Lambda_M\ T_M^{-1})^{-1} = T\ \Lambda_M^{-1}\ T_M^{-1}$$

As a consequence, $\Lambda_M + \Lambda_N$, $\Lambda_M\ \Lambda_N$, $z\ \Lambda_M$, $\Lambda^k$ are the eigenvalue matrices of $M + N$, $M\ N$, $z\ \Lambda_M$, and $M^k$, respectively. An algebraist would say that (1) **S** with the operations of matrix addition and scalar multiplication is a linear manifold, (2) **S** with the operations of matrix addition, matrix multiplication is a commutative ring, (3) there exists a linear ring isomorphism between **S** and the set of diagonal matrices with complex coefficients, (4) the set of elements of **S** that are invertible form a division ring, and (5) the ring isomorphism on **S** is a division ring isomorphism on the subset of invertible elements of **S** it is possible to prove that if each member of the set $N_1, N_2, \cdots, N_k$ commutes with $M$ so that we can diagonalize $N_k$ according to the scheme $N_k = T\Lambda_k T^{-1}$, and if $f = f(N_1, N_2, \cdots, N_k)$ is a rational function of its arguments, then $f$ commutes with $M$ and $f = Tf(\Lambda_1, \Lambda_2, \cdots, \Lambda_k)T^{-1}$.

**Continued fractions with complete commutative matrix coefficients**

The algebraic results from the previous sections can be specialized to build an algebra for the coefficients and approximants of such continued fractions. Once again consider the continued fraction $F$ from equation (1) with the matrix coefficients $a_k$, $b_k$, and $c_k (k = 1,2,3,...)$. Assume the matrix coefficients share a common complete set of eigenvectors. Any function of the matrix coefficients of $F$ that has a rational or power series expansion will commute with any of $a_k$,

For example, the continued fraction

$$\frac{I}{b_1 + a_2 \dfrac{I}{b_2 + a_3 \dfrac{I}{b_3 + \cdots} c_3} c_2}$$

commutes with $a_1$. A consequence is that $F$ can be rewritten

$$F = b_0 + \frac{I}{b_1 + a_2 \dfrac{I}{b_2 + a_3 \dfrac{I}{b_3 + \cdots} c_3} c_2} a_1 c_1$$

where the new first left partial numerator is equal to an identity matrix. If all the left partial numerators are brought to the right side of the fraction bars, then the resulting continued fraction is

$$F = b_0 + \cfrac{I}{b_1 + \cfrac{I}{b_2 + \cfrac{I}{b_3 + \cdots} a_3 c_3} a_2 c_2} a_1 c_1$$

in which all the left partial numerators are identity matrices. Thus, it is no longer necessary to distinguish between right and left partial numerators. In an abuse of notation it is possible to rewrite $F$ with simpler partial numerators as

$$F = b_0 + \cfrac{I}{b_1 + \cfrac{I}{b_2 + \cfrac{I}{b_3 + \cdots} c_3} c_2} c_1$$

The advantage obtained by working on continued fractions within an algebra of matrices with common eigenvectors is not just one of simplicity. The advantage lies instead in control over the eigenvalues and eigenvectors of the continued fraction and its approximants. For instance, $F$, the partial numerators, and the partial denominators share a complete set of eigenvectors. If $\beta_{kj}$ and $\gamma_{kj}$ are the eigenvalues of the $jth$ (common) eigenvector of $b_k$ and $c_k$, respectively, then the $jth$ eigenvalue of $F$ is

$$\lambda_j = \beta_{0j} + \cfrac{I}{\beta_{1j} + \cfrac{I}{\beta_{2j} + \cfrac{I}{\beta_{3j} + \cdots} \gamma_{3j}} \gamma_{2j}} \gamma_{1j}$$

Suppose, now, that Re$[\beta_{ij}] > 0$ and that $\gamma_{ij}$ is real and positive (i=1,2,3,...; j=1,2,...,n). By induction, it is easy to show that $Re[\lambda_j]$ non-negative and that the real parts of the $j$-th eigenvalues of each of the approximants of $F$ are positive. The stability of the finite difference approximation to the one-way wave equation will depend on positive-real properties, so the exercise is not a trivial one. The construction of the finite difference operator is taken up in the next chapter.