4.0 The Craft of Wave-Field Extrapolation

In this chapter we attend to those details which enable performance of a *high quality* job of downward continuing wave fields. From the point of view of the use of the seismic method to explore the earth, there will be no new concepts. There will, however, be some interesting examples of pitfalls. And in order to improve the quality of seismic images of the earth, several new and very interesting mathematical concepts will be introduced.

The Magic of Color

First we explore the two-dimensional spectral domain. In time series analysis we are familiar with the concept of filtering to enhance signal-to-noise ratio by suppressing some spectral regions and enhancing others. Spectral weighting can also be used on wavefields in the space of ω and k. In the absence of noise wave equation theory tells us what filters to use. Loosely, the wave equation is a filter with a flat amplitude response in (ω,k)-space and a phase response which corresponds to the time delays of ray tracing. Different regions of (ω,k)-space have different amounts of noise. But we need not display them all at the strength proposed by the wave equation, any more than we must process and display data with $\Delta x = \Delta z$.

Also, in our quest for quality, we must recall that we have made various approximations along our journey. Now is the time to see how this degrades our results, and to see how we can ameliorate the resulting difficulties. Three specific problems will be considered,

1 The frequency dispersion which results from the approximation of differential operators by difference operators

2 The anisotropy distortion of phase and group velocity which results from square-root approximations

3 The effect of truncation at the end of the survey line.

Briefly, we will also look at some stretching techniques which are used to facilitate use of Fourier methods.

Production Pitfall: Weak instability from v(x)

Some quality problems cannot be understood in the Fourier domain. Unless very carefully handled, lateral velocity variation can create instability.

As we have seen, the existence of lateral velocity jumps causes reflections from near-vertical faults. A more serious problem is that the extrapolation equations themselves have not yet been very carefully stated. The most accurate derivation so far has been done from dispersion relations which themselves imply velocity constant in x. We have never answered the question of how a dispersion relation containing vk_x^2 term should be represented. Should we use $v(x,z)\partial_{xx}$ or $\partial_x v(x,z)\partial_x$ or $\partial_{xx} v(x,z)$ or some combination? Each of these expressions implies a different numerical value for the internal reflection coefficient. Worse still, by the time all the axes are discretized it turns out that one of the most sensible arrangements of the term leads to reflection coefficients greater than unity and to numerical instability.

A weak instability is worse than a strong one. A strong instability will be noticed immediately. A weak instability might avoid notice, and ultimately lead to incorrect conclusions. A data processing anomaly could be mistaken for a geophysical anomaly. Such a mistake could be very costly. Weak instability is a production program with a built-in accident waiting to happen. Fortunately, the situation has now been corrected by means of stability analysis leading to a ***bullet-proof*** method.

cosmetic processes is not to be underestimated. On many occasions a comparison of processing techniques (for choice of contractor?) has been frustrated by accidental change in cosmetic parameters. Nor are cosmetic processes totally outside the world of wave-equation analysis. Indeed they can be made even more effective when built into a process rather than added on at the end.

Dip filtering often comes under suspicion as being a deceptive violator of the purity of real data. Indeed it can be misused to create events at will. But dip filtering does occur naturally and it also occurs as a by-product of various other processes. Our purpose here is to see how it can be used to advantage in a perfectly justifiable way.

False Semi-Circles in Migrated Data

A commonly missed opportunity is the failure to make effective use of dip filtering to suppress multiples. Without going into a detailed exposition of the theory and properties of focused multiple reflections, it can be stated that multiples are unlike primaries in one important respect. Their strength may change rapidly in the horizontal direction. They need not be spread out into broad diffraction hyperbolas as primaries must. This difference arises because they often spend much time focusing themselves in the irregular near-surface areas. Common evidence for this behavior is contained in the appearance of wide-angle migrated sections. Such sections often show semi-circular arcs coming all the way up to the surface. Such arcs are obviously not primary reflections. They can be multiples or unexplained impulsive noise. In either case they can be partially suppressed without touching primaries.

[1] SEP-25, pp 271-276.

4.1 Physical and Cosmetic Aspects of Wavefield Extrapolation[1]

Dip filtering and gain control are two processes whose purpose seems to be largely cosmetic, that is, the changes they make to the data are planned to improve its appearance. Criteria invoked to choose quantitative parameters of such processes are often vague and relate to human experience or visual perception. Objective criteria as signal and noise dip spectra are rarely used in a quantitative way. But the importance of

sion can lead to a consi[illegible]

Mixed Appearance of Dip-Filtered Data

An objection often raised against di[illegible] [illegible]y *mixed* is mea[illegible] a *mixed* appearance. [illegible]and that they ar[illegible] to have been averaged [illegible] at late times [illegible] true and it is inevitable [illegible]of reflection data [illegible]k as downward contin[illegible] izontal resolving power [illegible]sing lateral res[illegible]eases with depth. As [illegible]uation in two reasons for decrea[illegible]ppear. Second [illegible] cutoff along the evanescent [illegible] the down- high frequencies to dis[illegible]st increase as [illegible]e (ω,k_x)-space. Evanescent line izontal wavelengths m[illegible] It is unrealistic [illegible] wave propagation mo[illegible]y beyond found at greater depths. [illegible] channels sho[illegible] [illegible]countered. Such noise[illegible]el, and it tation and imagine th[illegible]ed appearance [illegible] total power at late times suppres- independence. If a mix[illegible] removing the [illegible] poses then I advoca[illegible]placing it by lo[illegible] generated noise and re[illegible] random noise.

[illegible] filtering is that it can [illegible]t that adjacent channe[illegible] give data [illegible]e no longer independe[illegible]ls appear [illegible] It is inevitable because[illegible]t. This is [illegible] decreases with time [illegible] the hor- [illegible]lution. First, dissipa[illegible] There are [illegible]even at constant freq[illegible]on causes [illegible]ays reach the higher [illegible]ency, hor- [illegible] to ignore this fundam[illegible] velocities [illegible] always have an appe[illegible]ntal limi- [illegible]s to be avoided for dis[illegible]arance of [illegible] low-velocity, coher[illegible]play pur- [illegible]-velocity, incoherent[illegible]t, signal- [illegible] Gaussian,

Accentuating Faults

It often happens tha[illegible]t the location of [illegible] the dominating effect [illegible]f stratified re[illegible] diffraction evidence of [illegible]faulting. This si[illegible] cess which weakens zer[illegible] and small dips, [illegible] 10° to 60° and then [illegible]presses the ve[illegible] energy. As with freq[illegible]ncy filtering, [illegible] because of the implied [illegible]long (and in sp[illegible] turns out that adjustab[illegible]e parameters [illegible] already available in the [illegible]45° equation.

[illegible]oil is controlled by fa[illegible] [illegible]ctors may overwhelm[illegible]ting. But [illegible]uation calls for a cosm[illegible] the weak [illegible] accentuates dips in th[illegible]etic pro- [illegible]y wide angles and e[illegible] range of [illegible]harp cutoffs are not [illegible]vanescent [illegible]ace, wide impulse res[illegible] desirable [illegible]at help achieve these [illegible]ponse. It [illegible] goals are

Zapping Multiples in D[illegible] Space

Think of the migr[illegible]tion of CDP sta[illegible] (ω,k_x,z)-space. Ordina[illegible]rily, velocity inc[illegible] ward continuation proc[illegible]eeds the velocity bites out more and m[illegible]ore area from t[illegible] this cutoff does not f[illegible]t the primary [illegible] should be suppressed a[illegible]s soon as it is en[illegible] [illegible]derable drop in t[illegible]

Decomposition of the 45 Degree Equation into Effects

There are various means of entering viscosity into wave propagati[illegible] theory. A well known means is to introduce a complex velocity into [illegible] ω^2/v^2 term of the scalar wave equation. This is much like introducing [illegible] complex ω. It may be recalled from Fourier transform theory that m[illegible] tiplication of a time function by a decaying exponential $\exp(-\alpha t)$ is [illegible] equivalent of replacing $-i\omega$ by $-i\omega+\alpha$ in the transform domain. [illegible] later section on impedance functions it is shown that replacing $-i\omega$ [illegible] by $(-i\,\omega/v)^\gamma$ describes the so-called "constant Q" absorption whi[illegible] accurately matches laboratory measurements.

Performing two iterations of the Muir square root expansion we g[illegible] an expression like

$$i V k_z^{(45)} \quad = \quad -i\,\omega_0 + \cfrac{X^2}{-i\,\omega_1 2 + \cfrac{X^2}{-i\,\omega_2 2}} \qquad (1)$$

In this expression X^2 denotes $V^2 k_x^2$ [or the positive definite mat[illegible] $(V\partial_x)(V\partial_x)^T = -V\partial_{xx} V$]. Previously, in expressions like (1) we ha[illegible] always written simply $-i\omega$, never expecting to want to make the d[illegible] tinction between $\omega_0, \omega_1, or\ \omega_2$. Indeed, we usually want each ω to ha[illegible] the same real part. However, by introducing different imaginary pa[illegible] we can introduce a viscosity which is angle dependent.

For example, we could choose each $-i\omega_j$ in (1) to be the consta[illegible] Q impedance function $(-i\omega)^\gamma$. The implied migration equation wo[illegible] then back out presumed frequency dissipation in the rocks. But t[illegible] would lead to a ridiculous enhancement of high frequencies. A bet[illegible] idea would be to keep ω_0 non-viscous but choose $-i\omega_1 = -i\omega_2 = (-i\omega)^\gamma$. With this idea there would be no attempt to back out the Q of the ver[illegible] cal path but there would be compensation of non-zero offsets to [illegible] zero-offset.

The choice of different real parts for the $-i\omega_j$ functions creates [illegible] amplification or attenuation which depends on dip. We could select t[illegible] real part of $-i\omega_1$ for the for-mentioned compensation of offset for [illegible] Then we could use the real part of $-i\omega_2$ for the purpose of suppressi[illegible] evanescent energy. It would be simple if we could choose the real part [illegible] $-i\omega_2$ so as to attenuate all dips above (i.e. velocities below) [illegible] medium's cutoff. What happens is almost as good but not quite as si[illegible] ple. Larry Morley discovered (SEP-16, p 109-119) that the absorpti[illegible] turns out to be extremely strong at 3/4 of the medium velocity bu[illegible] [illegible]

[illegible] so strong at lower velocities. (The 45° dispersion rela[illegible] [illegible]o at $Vk/\omega = 4/3$.) The width and depth of the absorp-tion curve hits ze[illegible] unfortunately, not controllable without messing up the migration, but tion is somewhat [illegible] velocity noise is better eliminated by some other massive very lo[illegible]ry. means. In summa[illegible]

	[illegible] ain effect of terms in the 45° equation			
	Real Part		Imaginary Part	
term	(influences traveltime)	[illegible] conversion	cos[illegible]metic filter	physical absorption
ω_0	time/depth [illegible]	[illegible]acking vel.	fault e[illegible]hancement	Q-offset compensation
ω_1	migration/s[illegible]	[illegible]acking vel.	steep [illegible]p suppress	evanescent junk
ω_2	migration/st[illegible]			

Dip Filtering Too[illegible]

Gain Control Does [illegible]

If data is expo[illegible]nentially gained upward before migration, then hyper-bola flanks are bo[illegible]osted in strength before migration moves them in with the hyperbola top. This is certainly a dip-enhancement feature. Let us consider this more specifically. Take the Z-transform of a time func-tion a_t.

$$A(Z) \quad = \quad a_0 + a_1 Z + a_2 Z^2 + \cdots$$

We now define the exponentially gained time function by

$$\uparrow A(Z) \quad = \quad a_0 + a_1 e^{\alpha} Z + a_2 e^{2\alpha} Z^2 + \cdots$$

The symbol $\uparrow$ is indicative of the exponential gain. Mathematically $\uparrow$ means that the Z is replaced by $e^{\alpha}Z$. Consider a polynomial multipli-cation or convolution of time functions

$$C(Z) \quad = \quad A(Z)\,B(Z)$$

Obviously,

$$\uparrow C \quad = \quad (\uparrow A)\,(\uparrow B)$$

[illegible]n do exponential gain either before or after convolu-

This means we ca[illegible] downward-continuation operator $\exp(+i\,k_z\,z)$ for some tion. [illegible]e fixed k_x. It is a function of ω which may be

Think of the [illegible] fixed z and so[illegible] cess by a cosmetic

before stack. Tr

the steepest dip

$$A(Z) \quad = \quad a_0 + a_1 \frac{1}{Z} + a_2 \frac{1}{Z^2} + \cdots$$

gain control ca

correctly eliminating the coefficients of positive powers of Z is associ-
Perhaps it was ...shing negative powers. So $\uparrow A$ has a weakened precur-
or whether the ...ttenuate flanks rather than moving them; and it is said
tial filtering?

...sical point of view dictates that cosmetic functions like
...dip filtering should be done after processing, say $\uparrow(AB)$.

The Substitutio...

...lent to $(\uparrow A)(\uparrow B)$, which means using a viscous operator
The upward ... gained data. In practice it is common to forget the
$Z \to Ze^{\alpha}$. The ...eate $A(\uparrow B)$. I like such cosmetics in the sense that I
$\uparrow C = (\uparrow A)(\uparrow B)$. ...ents carry more information than flat ones. But going
tion for Z, no... attenuation seems to be preferable. The decomposition
trivial substitu...ion into the three main parts gives much flexibility for
compression ... these goals.
by two. Anothe...

...oherence or Rejection by Filtering?

...oid the pitfall of judging a supposed non-cosmetic pro-
...ic effect. I once got caught. The process was migration
...e feature deemed desirable was the relative strength of
...r event on the record, a fault plane reflection. But even
... affect dip spectra! I hoped the process was working by
...ating some of the rejection of steep dips by CDP stack.
...but how could I know whether this was really happening
...rocess had an accidental ability to enhance dips by spa-

...Operator

... $\uparrow$ operator has been defined to mean the substitution
...main property of this operator is that if $C = AB$ then
This property would be shared by any algebraic substitu-
... just the one for exponential gain. Another relatively
...ion may be used to achieve time-axis stretching or
...or example replacing Z by Z^2 stretches the time axis
expressed in the ... substitution which has a considerably deeper meaning
move *upward* or
cate by

Exponentially ...
ated with dimin...
sor, it tends to ...
to be viscous.

A purely phy...
gain control and
But this is equiv...
on exponentially
viscosity and cr...
think dipping ev...
beyond 45° dip, ...
of the 45° equat...
reaching toward

Rejection by the ... time domain as a filter a_t. But the hyperbola flanks
We should a... migration. So the filter is *anticausal* which we can indi-

than either of the previous two is t...
sipation operator $(-i\omega)^{\gamma}$. In sum...

Substitutions for ... [*all preserve* ...]	
Exponential Gr...	
Time expansion (...	
(Inverse) Constant Q ...	

...he substitution of the constant Q dis-
...nary,

...Z-transform variable Z [$C(Z)=A(Z)B(Z)$]	
...wth	$Z \to Ze^{\alpha}$ ($i\omega \to i\omega + \alpha$)
...$\alpha > 1$)	$Z \to Z^{\alpha}$
...dissipation	$-i\omega \to (-i\omega)^{\gamma}$

from the 45° ex
These are depict
and the heart ar
zone, and the x-
the aliasing freq
(x,t)-plane when
ing in figure 2 a

...rsion and Wave-Migration Accuracy[1]

...ypes of errors in wave migration. Of greatest ...quency dispersion, occurring when different ... different speeds. This may be reduced by ... of finite difference approximations to ...e cure is sufficient refinement of the ...condary importance, and the topic of the ...y dispersion. This occurs when waves pro... ...ions do so at different speeds. It is remedied ...xpansion. As a practical matter, it is rarely ... well-known 45° equation. It is not that we do ...er angles; we often do. The problem at large ...dge of seismic velocity is seldom, if ever, ...tify the trouble involved in using wide angle

...y associated with propagation of light in cry... ...ology anisotropy is occasionally invoked to ...ween borehole velocity measurements (verti... ...ty determined by normal moveout (horizontal ...arise as an undesirable side effect in seismic ...ssing. This subject was analyzed in detail in ...e restated here.

4.2 Aniso...

There are tw... practical import... frequencies pro... improving the ... differentials. ... differencing ... present study, ... pagating in diffe... by the Muir squ... necessary to go ... not observe wave... angles is that ... sufficiently accu... equations.

...m a Huygen's secondary source is a semicir... that actually results from the 15° extrapola... The secondary source that actually results ... equation is an interesting, heart-like shape. ...e 1. In practice the top parts of the ellipse ...bserved because they are in the evanescent ...dom refined sufficiently for them to be below ... center of the heart is sometimes seen in the ...ogram is used. It is depicted by a line draw... a 45° diffraction program in figure 3.

Anisotropy is ... stals. In reflect... explain the discre... cal propagation) ... propagation). It ... calculation and ... SEP-8 and the res...

Point Source Res...

The ideal wav... cle. The seconda... tion equation is ...

1 SEP-20, p 207-220

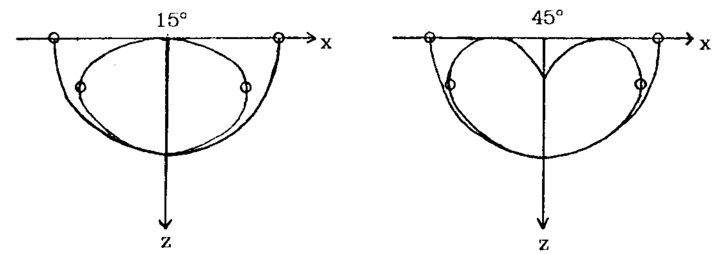


FIG. 1. Wavefronts of 15° and 45° extrapolation equations, inscribed within the exact semicircle. Waves with $\sin\vartheta = vk_x/\omega = \pm 1$ are marked with small circles.

Wave Front Direction and Energy Velocity

In wave propagation we are familiar with the idea of energy propagating perpendicular to a wavefront. When there is anisotropy dispersion the two directions differ. The apparent horizontal velocity seen along the surface is dx/dt. The apparent velocity along the vertical, seen in a borehole, is dz/dt. Because of geometrical considerations, both of these apparent speeds exceed the wave speed. A vector perpendicular to the wavefront with a magnitude inverse to the velocity is called the slowness vector:

$$\textit{slowness vector} \quad = \quad \left(\frac{dt}{dx} , \frac{dt}{dz} \right)$$

A vector perpendicular to the wavefront scaled to the speed of the wavefront is evidently the slowness vector divided by its squared magnitude. It is called the phase velocity vector:

$$\textit{phase velocity} \quad = \quad \frac{\left(\frac{dt}{dx} , \frac{dt}{dz} \right)}{\left(\frac{dt}{dx} \right)^2 + \left(\frac{dt}{dz} \right)^2}$$

t

FIG. 2. 45° heart theory.

$$\textit{slowness vector} \quad = \quad \left(\frac{k_x}{\omega} , \frac{k_z}{\omega} \right) \qquad (1a)$$

The direction of energy propagation is somewhat more difficult to derive, but it turns out to be the so-called *group velocity*.

$$\textit{group velocity} \quad = \quad \left(\frac{\partial}{\partial k_x} , \frac{\partial}{\partial k_z} \right) \omega(k_x, k_z) \qquad (1b)$$

For the scalar wave equation $\omega^2/v^2 = k_x^2 + k_z^2$, the group velocity and the slowness vector are in the same direction, as may be verified with equation (1). The most familiar type of dispersion is frequency dispersion, where different frequencies travel at different speeds. It is shown in SEP-8 that the familiar (15°, 45°, etc.) extrapolation equations do not exhibit frequency dispersion. Specifically, as functions of ω and k_x/ω, the velocities do not depend on ω. In other words, the elliptical and heart shapes of figure 1 are not frequency-dependent.

The most interesting aspect of anisotropy dispersion is that energy appears to be going in one direction when it is actually going in another. To illustrate this phenomenon we will consider an exaggerated instance

In a disturbance with sinusoidal form, the phase may be set equal to a constant, and the derivatives may be determined, giving

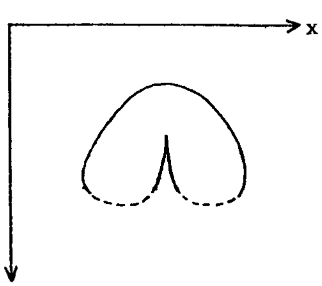
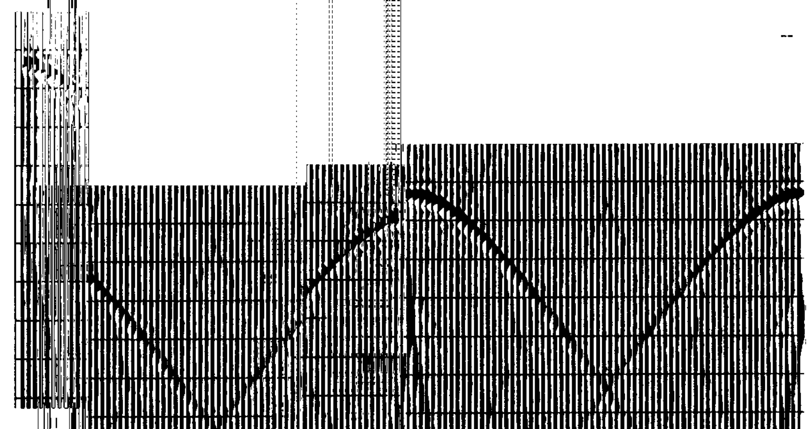


FIG. 3 ... example. (Morley) 45° heart ...

of it ... has a downward component and the phase ... velocity ... component. Figure 4 depicts the dispersion ... n which the group ... equation. A slowness vector, which is ... velocity has an up ... extrapolation ... ected by drawing an arrow from the origin ... relation of the 45° ... been ... e corresponding direction of group velocity ... wavefront normal, has ... curve. ... aphically by noting that group velocity is ... to the dispersion ... mined ... ator in equation (1b). Think of ω as the ... y may now be determined ... north and k_x points east. Then the ... defined by the grad ... points ... onstant altitude. Different numerical ... a ... ight of a hill where ... tour of ... wing figure 4 to different scales. The Figure ... sion relation is a contour ... from ... he gradient, is perpendicular to the ... of frequency result ... ction of ... frame ... velocity, in the dire ... larger ... urs of constant ω ...

... menon is most clearly recognized in appear ... ion phen ... ood on a single frame, as in figure 5. The ... e anisotropy dispers ... understanding energy flow from the top, through ... e, although it can ... interpret ... gle, reflecting from the side of the 5 or 6 ... 6 is a line drawing ... he 45° ... a of the figure which is sufficiently ... ism, reflecting at ... g an ... e fronts to be recognized as energy ... and finally enter ... the phase ... actually propagating downward. ... and uncluttered for ... ward but ... ... ntly propagating ... tion can propagate upward in figures ... r information ... nt of the program which calculates ... at neither energy no ... stand ... is obvious from the ...



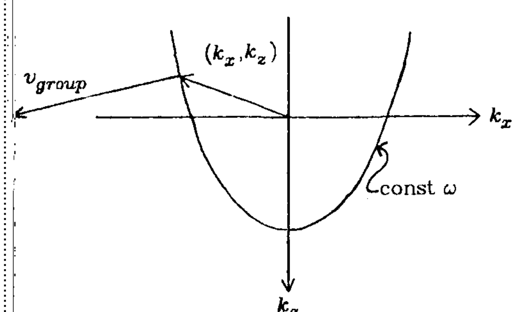


FIG. 4. Dispersion relation for downgoing extrapolation equation showing group velocity vector and slowness vector (perpendicular to wavefront).

figure 5. The program does not have the entire frame in memory; it produces one horizontal strip at a time from the strip just above. Thus the movie's phase fronts, which appear to be moving upward, seem very curious. Theoretically we do not expect wave extrapolation, particularly the 45° equation, to handle angles to 90°. Yet the example shows that these extreme cases are indeed handled, although in a somewhat perverted way.

I once observed a similar circumstance on reflection seismic data from a geologically overthrusted area. The data could not be made available to me at the time, and by now it is probably long lost in the owner's files, so you will have to be content with the recollected line drawing in figure 7. The increasing velocity with depth causes the ray to bend upward and reflect from the underside of the overthrust. To see what is happening in the wave equation, it is helpful to draw the dispersion curve at two different velocities, as in figure 8. Downward continuation of a bit of energy with some particular stepout $dt / dx = k_x / \omega$ begins at a quite ordinary angle on the near surface, slow velocity dispersion curve. But as deeper velocity material is encountered at depth, that same stepout implies a negative phase velocity. The situation resembles that in figures 5 and 6. Although the thrust angle is unlikely to be quantitatively correct, the general picture is appropriate.

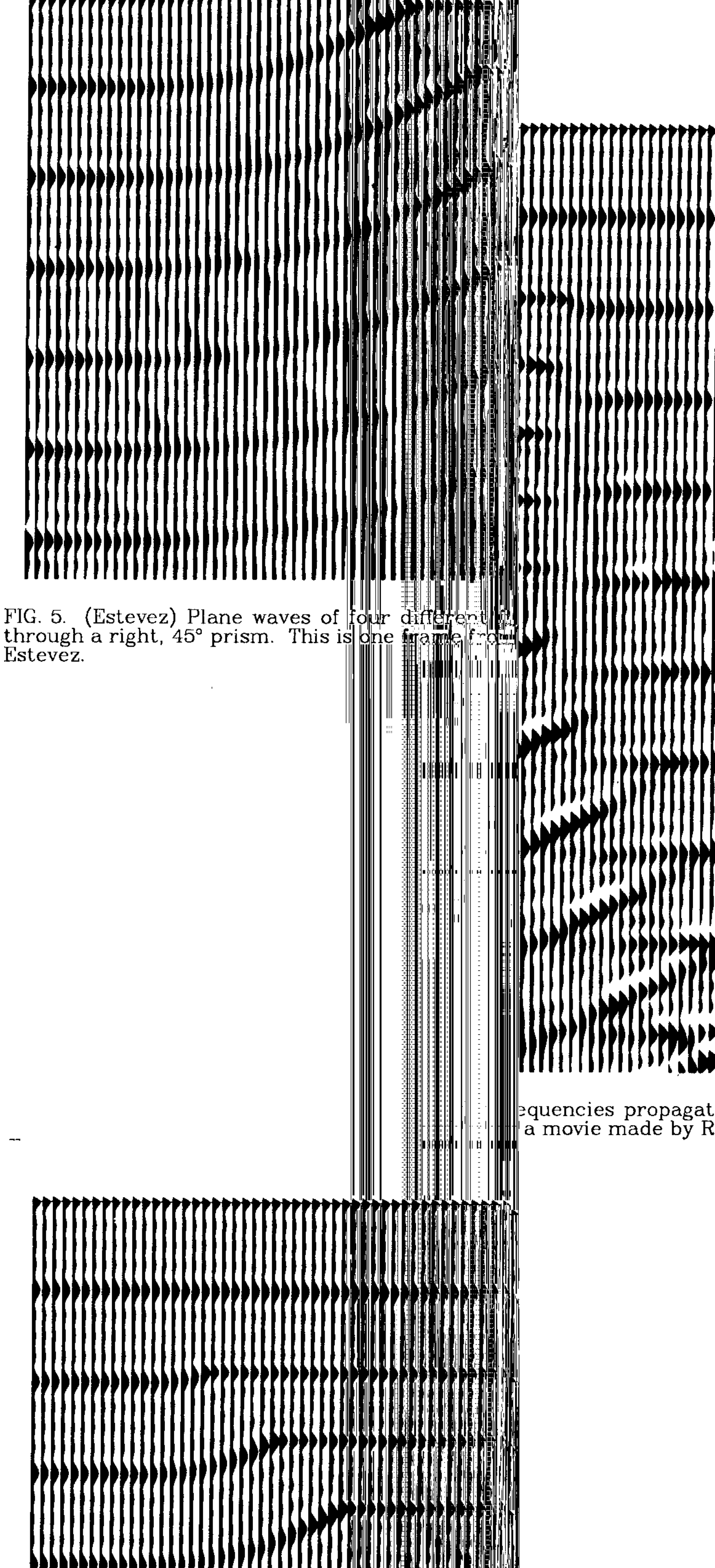


FIG. 5. (Estevez) Plane waves of four different frequencies propagating through a right, 45° prism. This is one frame from a movie made by Raul Estevez.

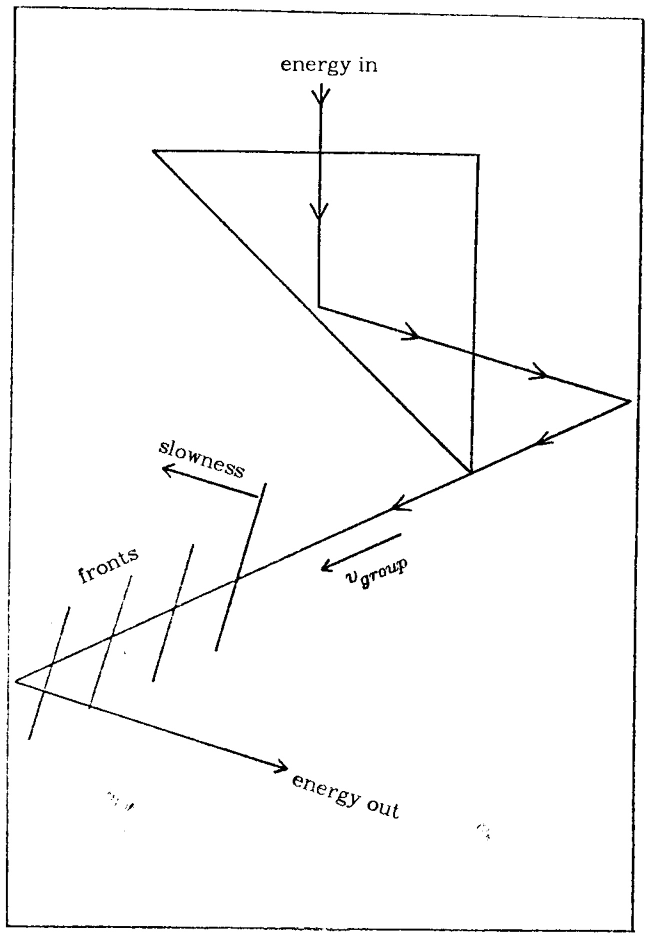


FIG. 6. Interpretation of some energy flow in figure 5 which illustrates different directions of energy and wavefront normal.

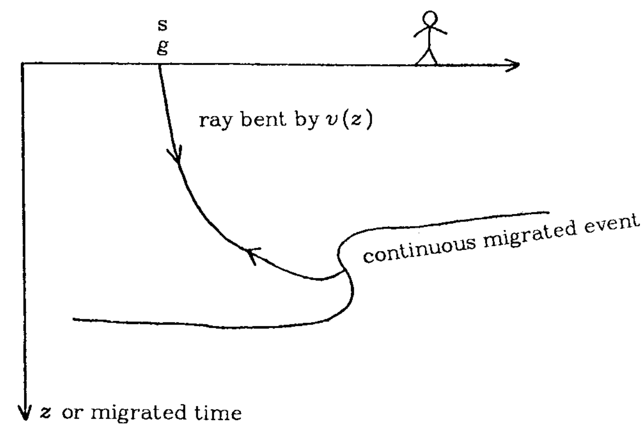


FIG. 7. Ray reflected from underside of overthrust.

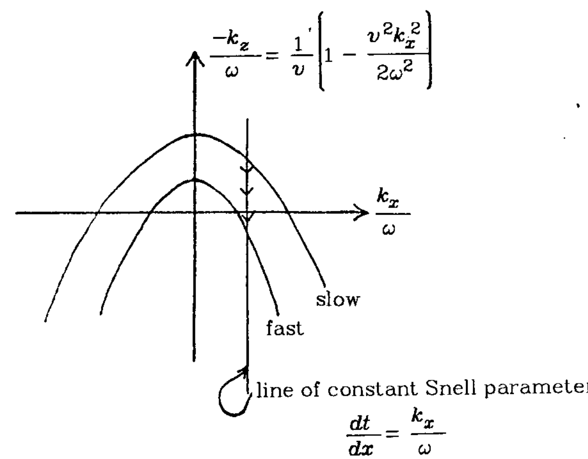


FIG. 8. Dispersion curve at two different velocities v_{fast} and v_{slow}.

Analyzing E[illegible]rrors of Migration

g reflector that is flat and regular may be analyzed in its
ı the phase velocity concept. The group velocity concept is
y when more than one angle is simultaneously present, as
ase when we analyze the point scatterer response. In addi-
ng bed with reflection amplitude variable along the bed must
with group velocity. Figure 9 depicts a smooth, flat, dipping

A dipp[illegible]as been undermigrated because the $\hat{k}_z$ defined by some
entirely wi[illegible]are root approximation or some numerical approximation
required o[illegible]h the correct square root value of k_z.
will be the
tion, a dipp
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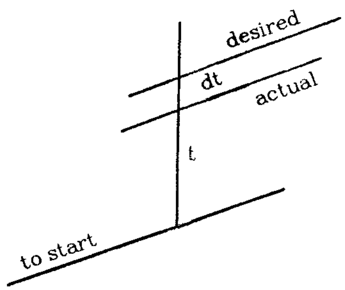

FIG. 9. Und[illegible]rmigrated dipping reflector.

[illegible]r in this case is entirely a time shift error. Since in this case
[illegible]ken the reflection coefficient to be constant along the
[illegible] lateral shift error can be recognized. The time error may
[illegible]ally determined by

$$\frac{dt}{t} \approx \frac{\hat{k}_z - k_z}{k_z} \tag{2}$$

The erro
we have ta[illegible]alled 15° equation it turns out that about a half-percent
reflector, n
be theoretic

For the sq

[illegible]t 25°.

phase error is made [illegible]ne the error in the collapse of a hyperbola. Figure
[illegible]ward continuation of a hyperbola. For clarity the
[illegible]on was not taken all the way to the focus. We will

Next, we determ[illegible]of some Snell's parameter $p = dt/dx$ by selecting
10 depicts the dow[illegible]onstructing a tangent line segment of slope p to
downward continua[illegible]ids. If there were a little amplitude anomaly where
keep track of a ray [illegible]would be able to identify it on each of the hyper-
some slope p and [illegible]elocity is needed because either a curved event or
each of the hyperbol[illegible]ly requires a range of plane wave angles to be
the slope is p, you [illegible] analogous to a time series wavelet's need for a
boloids. The group [illegible]to represent it. You will notice in figure 9 that the
an amplitude anom[illegible] moved is too little; likewise, the actual lateral dis-
represented. This [illegible]small; so in practice, the errors are sometimes com-
range of frequencies [illegible]t a six percent increase of either z or v. The
actual amount of tim[illegible] errors are shown to be
tance moved is too s
pensated for by abo
actual amounts of the

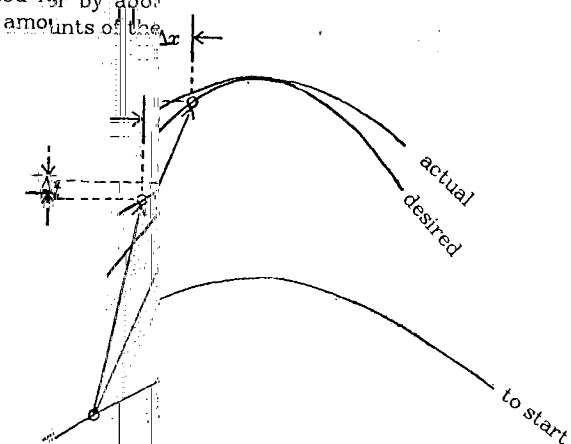

FIG. 10. Error of hype[illegible]rbola collapse. Note that the actual curve is above
the desired curve, bu[illegible] the actual point is below the desired point.

$$\frac{\Delta t}{t} = \frac{\frac{\partial}{\partial \omega}(\hat{k}_z - k_z)}{\frac{\partial}{\partial \omega} k_z} \tag{3a}$$

$$\frac{\Delta x}{x} = \frac{\frac{\partial}{\partial k_x}(\hat{k}_z - k_z)}{\frac{\partial}{\partial k_x} k_z} \tag{3b}$$

where k_z is taken to be a function of ω and k_x. It turns out that for the 15° equation, about a half-percent group velocity error occurs at 20°. Thus the group velocity error is generally worse than the phase velocity error.

Derivation of Group Velocity Equation

We can make up an impulse function at the origin in (x,z)-space by superposing Fourier components:

$$\int\int e^{+ik_x x + ik_z z}\, dk_x\, dk_z \tag{4}$$

Physics and possibly numerical analysis lead to a dispersion relation which is a functional relation between ω, k_x, and k_z, say, $\omega(k_x,k_z)$. The most common example is the scalar wave equation $\omega^2 = (k_x^2 + k_z^2)/v^2$. The solution to the equations is

$$e^{-i\omega t + ik_x x + ik_z z} \tag{5}$$

Integrating (5) over (k_x,k_z) will produce a monochromatic time function which at $t=0$ is an impulse at $(x,z)=(0,0)$. The expression at some very large time t is

$$\int\int e^{-it[\omega(k_x,k_z) - k_x x/t - k_z z/t]}\, dk_x\, dk_z \tag{6}$$

At t very large, the integrand is a very rapidly oscillating function of unit magnitude. Thus the integral will be nearly zero unless we can get the quantity in square brackets to become nearly independent of k_x and k_z for some sizable area in (k_x,k_z)-space. To find such a flat spot we proceed as if we were finding the max or min of a two-dimensional function, that is, by setting derivatives to zero. This analytical approach is known as the stationary phase method:

$$0 = \frac{\partial}{\partial k_x}[\;] = \frac{\partial \omega}{\partial k_x} - \frac{x}{t} \tag{7a}$$

$$0 = \frac{\partial}{\partial k_z}[\;] = \frac{\partial \omega}{\partial k_z} - \frac{z}{t} \tag{7b}$$

So in conclusion, at time t the dist…urbances will be located at

$$(x,z) = t\left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_z}\right) \tag{8}$$

which justifies the definition of gro… p velocity.

Derivation of Energy Migration Eq… ation

Energy migration in (x,t)-spa… e is analyzed in a fashion similar to the derivation of group velocity. You take depth to be large in the integral

$$\int\int e^{iz[k_z(\omega,k_x) \cdots \omega t/z + k_x x/z]}\, d\omega\, dk_x \tag{9}$$

The result is that the energy goes t…

$$(x,t) = z\left(-\frac{\partial k_z}{\partial k_x}, \frac{\partial k_z}{\partial \omega}\right) \tag{10}$$

This justifies our previous asserti… on that (3) can be used to analyze energy propagation errors.

4.3 Frequency Dispersion and Wave-Migration Accuracy[1]

Frequency dispersion is a result of different frequencies propagating at different speeds. The physical phenomenon of frequency dispersion is rarely heard in daily life, although many readers may have heard it while ice skating on lakes and rivers. Elastic waves caused by cracking ice propagate dispersively changing pops into percussive notes. Even while frequency dispersion is a barely perceptible phenomenon in reflection seismology, it is a substantial nuisance in seismic data processing, where we often see artful displays of the difference between differential operators and difference operators. As such, it is an embarrassment to process builders. Dispersion is not an essential feature of data processed by finite differences: it can always be suppressed by sampling more densely, and it is the job of the production analyst to see that this is done. Figure 1 depicts some dispersed pulses.

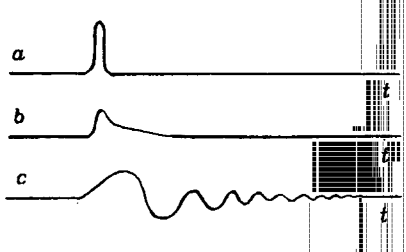

FIG. 1. (a) A pulse. (b) A pulse slightly dispersed as by the physical dissipation of high frequencies. (c) A pulse with a substantial amount of frequency dispersion, as could result from careless data processing.

But dispersion can be a useful warning to the seismologist that the data itself is in danger of transgressing the boundary into aliased space. Frequency-domain methods do not depend on difference operators so they have the advantage that they do not show dispersion. Penalties for this advantage are: (1) limitation to constant material properties, (2) wraparound, (3) occurrence of spatial aliasing without the warning of dispersion.

[1] SEP-24, p 299-309.

Spatial Aliasing

Aliasing can occur on the axes of time, depth, geophone, shot, midpoint, offset, or crossline. We will begin on the horizontal space axis where the problem is worst. The dispersion relation of the wave equation enables us to compute the vertical spatial frequency k_z from the temporal frequency ω, the velocity v, and the horizontal spatial frequency k_x by the semicircle relation $k_z(\omega, k_x) = \sqrt{\omega^2/v^2 - k_x^2}$. Sampling on the x-axis gives an upper limit to k_x equal to the Nyquist frequency $\pi/\Delta x$. Both frequency-domain methods and finite-difference methods treat higher frequencies as if they were folded at the Nyquist frequency. Thus the semicircle dispersion relation is replicated above the Nyquist frequency as shown in figure 2.

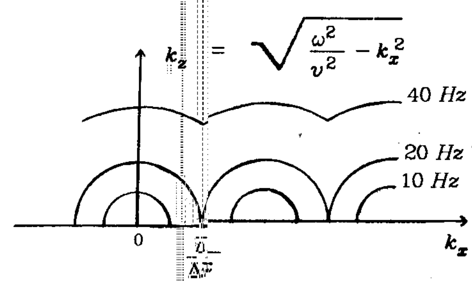

FIG. 2. The effective dispersion relation of the wave equation when the horizontal axis is sampled. Values are given for zero-offset migration where $\Delta y = 25$ meters and velocity $v = 2000 m/sec$. The semicircular arcs correspond to frequencies of 40, 20, and 10 Hz.

The problem of spatial aliasing begins when two circles touch each other, as is shown at 20 Hz in figure 2. This occurs when a half wavelength $v/2f = \pi v/\omega$ equals the spatial sample rate Δx. We must be sure to use half the rock velocity or else we need to consider the two way travel time. Thus the aliasing problem is avoided if $2f \Delta x < 1/2\, v_{rock}$. For rock velocity equal 2 km/sec safe frequencies are below those in the table.

	Δx	frequency
standard	25m	20 Hz
reconnaissance	50m	10 Hz
3-D cross line	100m	5 Hz

Another view is that steeply dipping waves are suppressed by the geophone group. (This disregards shot-space aliasing.) In this view the limitation should be thought of in terms of angles at which energy is missing from the data. Taking the ray angle to be 30° instead of 90° doubles horizontal wavelengths. Thus for 30° safety from aliasing at rock velocity of 2km/sec, we need frequencies less than

	Δx	frequency
standard	25m	40 Hz
reconnaissance	50m	20 Hz
3-D cross line	100m	10 Hz

Some perspective on the significance of wide- angle processing is gained by realizing that data commonly exhibit good signals above 40 Hz.

Second Space-Derivatives

The defining equation for a second-derivative operator is

$$\frac{\delta^2}{\delta x^2} P \quad = \quad \frac{P(x+\Delta x) - 2P(x) + P(x-\Delta x)}{(\Delta x)^2} \tag{1}$$

The second derivative operator is defined by taking the limit

$$\frac{\partial^2}{\partial x^2} P \quad = \lim_{\Delta x \to 0} \quad \frac{\delta^2}{\delta x^2} P \tag{2}$$

Many different equations can all go to the same limit as Δx goes to zero. So the problem is to find an expression which is accurate when Δx is larger than zero. The practical problem is to find an accurate expression which is not too complicated. Our first objective is to see how to evaluate quantitatively the accuracy of equation (1). Second, we will look at an expression that is slightly more complicated but much more accurate.

The basic method of analysis is Fourier transformation. More simply, we take derivatives of the complex exponential $P = P_0 \exp(ikx)$ and look at errors as a function of the spatial frequency k. For the second derivative we have

$$\partial_{xx} P \quad = \quad \frac{\partial^2}{\partial x^2} P \quad = \quad -k^2 P \tag{3}$$

We define $\hat{k}$ by an analogous expression with the difference operator

$$\delta_{xx} P \quad = \quad \frac{\delta^2}{\delta x^2} P \quad = \quad -\hat{k}^2 P \tag{4}$$

Ideally $\hat{k}$ would equal k. Inserting the complex exponential into (1) we get an expression for $\hat{k}$ in terms of k

$$-\hat{k}^2 P \quad = \quad \frac{P_0}{\Delta x^2}\left[e^{ik(x+\Delta x)} - 2e^{ikx} + e^{ik(x-\Delta x)} \right] \tag{5a}$$

$$-\delta_{xx} \quad = \quad \hat{k}^2 \quad = \quad \frac{2}{\Delta x^2}[1 - \cos(k\Delta x)] \tag{5b}$$

It is a straightforward matter to make plots of $\hat{k}\Delta x$ versus $k\Delta x$ from (5). The half-angle trig formula allows an analytic square root of (5) which is

$$\hat{k}\,\Delta x \quad = \quad 2 \sin \frac{k\,\Delta x}{2} \tag{5c}$$

Series expansion shows that for low frequencies $\hat{k}$ is a good approximation to k. At the Nyquist frequency, defined by $k\,\Delta x = \pi$, the approximation $\hat{k}\,\Delta x = 2$ is a poor approximation to π.

The 1/6 Trick

Increased absolute accuracy may always be purchased by reducing Δx. Increased accuracy relative to the Nyquist frequency may be purchased at a cost of computer time and analytical clumsiness by adding higher order terms, say

$$\frac{\partial^2}{\partial x^2} \quad \approx \quad \frac{\delta^2}{\delta x^2} - \frac{\Delta x^2}{12}\frac{\delta^4}{\delta x^4} + etc. \tag{6}$$

As Δx tends to zero (6) tends to the basic definition (1) and (2). Coefficients like the 1/12 in (6) may be determined by the Taylor-series method if great accuracy is desired at small k. Or a somewhat different coefficient may be determined by curve-fitting techniques if accuracy is desired over some range of k. In practice I believe (6) is hardly ever used because there is a less obvious expression which offers much more accuracy at less cost! The idea is indicated by

$$\frac{\partial^2}{\partial x^2} \approx \frac{\dfrac{\delta^2}{\delta x^2}}{1 + \dfrac{\Delta x^2}{6}\dfrac{\delta^2}{\delta x^2}} \tag{7a}$$

The accuracy may be numerically evaluated by substituting from (5) to get

$$\left(\frac{\hat{k}\,\Delta x}{2}\right)^2 = \frac{\sin^2 \dfrac{k\,\Delta x}{2}}{1 - \dfrac{1}{6} 4 \sin^2 \dfrac{k\,\Delta x}{2}} \tag{7b}$$

The square root is plotted in figure 3.

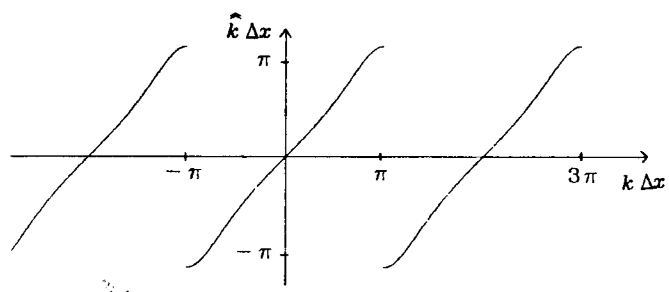

FIG. 3. (Hale) Accuracy of the second-derivative representation of (7) as a function of spatial wavenumber. The sign of the square root of (7b) was chosen to agree with k in the range $-\pi$ to π and to be periodic outside the range.

If the 1/6 in (7) were replaced by 1/12 then (7) and (6) would agree to second order in Δx. Actually the 1/12 comes from series expansion but the 1/6 fits over a wider range and is a value in common use. F. Muir pointed out that the value $(1/\pi^2 - 1/4) \approx 1/6.7$ gives an *exact* fit at the Nyquist frequency and a quite accurate fit over all lower frequencies!

Indeed, in 1980 it may be said that few explorationists consider the accuracy deficiency of (7) to be large enough to warrant interpolation of field-recorded values.

Rather than pursue this further to provide a thorough analysis of errors, let us be sure it is clear how (7) is put into use. The simplest prototype equation is the heat-flow equation.

$$\frac{\partial}{\partial t} q = \frac{\partial^2}{\partial x^2} q \approx \frac{\delta_{xx}}{1 + \dfrac{\Delta x^2}{6}\,\delta_{xx}}\, q \tag{8}$$

Just multiply through the denominator thus

$$\left[1 + \frac{\Delta x^2}{6}\delta_{xx}\right]\frac{\partial}{\partial t} q \approx \delta_{xx}\, q \tag{9}$$

Time-Derivatives and the Bilinear Transform

You might be inclined to think that a second derivative is a second derivative and that there is no mathematical reason to do time-derivatives differently than space-derivatives. This is wrong. A hint of fundamental disparity is found by considering boundary conditions. With time-derivatives (and often with the depth z-derivative) we generally have a concept of causality, which means that the future is determined solely from the present and past. Appropriate boundary conditions on the time axis are initial conditions - that is, specification of the function (and perhaps some derivatives) at *one* point, the initial point in time. For depth z that special point is the earth surface at $z=0$. But lateral space-derivatives are different. They require boundary conditions at two widely separated points, usually at the left and right sides of the volume under consideration.

Causal differentiation is a deep subject. Its importance to stability in wave analysis merits more lengthy consideration in a later chapter on advanced wave extrapolation. But we have already seen the main idea, which is embedded in the Crank-Nicolson differencing scheme. It remains to examine accuracy and frequency dispersion.

Begin with a time function p_t. We define its Z-transform by

$$P(Z) = \cdots p_{-2}Z^{-2} + p_{-1}Z^{-1} + p_0 + p_1 Z + p_2 Z^2 + \cdots \tag{10}$$

Define an operator $-i\hat{\omega}\Delta t$ by

$$\frac{1}{-i\hat{\omega}\Delta t} \quad = \quad \frac{1}{2}\,\frac{1+Z}{1-Z} \qquad (11)$$

…us apply this operator on P to get the Z-transform … Q of another function q_t.

Let … time …

$$Q(Z) \quad = \quad \frac{1}{2}\,\frac{1+Z}{1-Z}\,P(Z) \qquad (12a)$$

…ply both sides by $(1-Z)$:

$$(1-Z)\,Q(Z) \quad = \quad \frac{1}{2}(1+Z)\,P(Z) \qquad (12b)$$

Multi… …te the coefficient of Z^t on each side:

$$q_t - q_{t-1} \quad = \quad \frac{p_t + p_{t-1}}{2} \qquad (12c)$$

Equa… …ng p_t to be an impulse function we see that q_t turn… s out to be a …function, that is,

$$p \quad = \quad \cdots 0,0,1,0,0,0, \cdots \qquad (13a)$$

$$q \quad = \quad \cdots 0,0,\frac{1}{2},1,1,1, \cdots \qquad (13b)$$

Tak… step… …y to time t. … expressions …) represents … accuracy of … substituting

t approximates the integral of p_t from minus infinit… …g $Z = \exp(i\omega\Delta t)$ equations (10), (11), and (12a,b) are … e Fourier transform domain and the operation of (1…

So … …rical integration by the Crank-Nicolson method. The … Taki… integration (or differentiation) is evaluated by … in th… exp$(i\omega\Delta t)$ into (11), say … nume… the … $Z = e$…

$$-i\hat{\omega}\Delta t \quad = \quad 2\,\frac{1-e^{i\omega\Delta t}}{1+e^{i\omega\Delta t}} \quad = \quad 2\,\frac{e^{-i\omega\Delta t/2} - e^{i\omega\Delta t/2}}{e^{-i\omega\Delta t/2} + e^{i\omega\Delta t/2}} \qquad (14)$$

$$= \quad -2i\,\frac{\sin(\omega\Delta t/2)}{\cos(\omega\Delta t/2)}$$

$$\frac{\hat{\omega}\Delta t}{2} \quad = \quad \tan\frac{\omega\Delta t}{2}$$

… of approxi- …son method. …ro. Relative …30%, 3%, and … of small Δt … there does … which is any

Equa… …ion (14) is the fundamental statement of the accurac… mati… …n of the first-derivative operator by the Crank-Nicol… Serie… …s expansion shows that $\hat{\omega}$ goes to ω as Δt goes to ze… error… …s in ω at (4, 10, and 20) points per wavelength are (… 1%). These errors are quite large, calling for either a choice or a … more accurate method than (14). The bad news is that … not s… …seem to exist a representation of causal differentiation …

more accura… the 1/6 trick … …e than Crank-Nicolson. In other words we have nothing like interval Δt … for time-derivatives. So we must reduce the time sample … considerably from the Nyquist criterion.

On the o… …ther hand most geophysical differential equations have time-inva… …t coefficients so we can solve them in the ω-domain rather than in the t… …me domain. Or we might find we do not need to have *causal* time-deriva… …es: antisymmetric derivatives might do. With the depth z-axis we are … largely stuck with causal derivatives, although we could use Fourier … methods over layers. But the depth axis is not so trouble- some as the x- and t-axes because it usually affects computer time only, not data storage.

The pract… …cal picture may not be as dreary as the one I am painting. Many people a… …re very pleased with both the speed and accuracy of time- domain mig… …tions at $\Delta t = 4$ milliseconds.

Accuracy – The Contractor's View

A chain is … said to be no stronger than its weakest link. Economy dic- tates that the links should all be made of equal strength. Likewise, in the constru… …on of a production program for wave-equation migration, weakness ar… …es from approximations made in many different places. Again, econom… …y dictates that funds to purchase accuracy should be dis- tributed to w… …ere they will do the most good. Geophysical contractors naturally bec… …me experts on accuracy-cost trade-offs in the migration of stacked data. Certain broader questions also merit study, such as the error associate… …d with velocity uncertainty and the error associated with migration afte… stack rather than before. Here we will just look at some of the more o… …vious considerations.

Migration … is basically just a process of downward extrapolating sur- face data. A… …he various approximations imply timing errors, which for a single freque… …ncy are just phase errors. It is easy to write equations for these phase e… …rrors. The true phase ζ at a depth z is given by the integral of k_z … with depth. But we may as well replace the integral by the average in… …tegrand times the depth:

$$\zeta_{true} \quad = \quad z\,k_z \qquad (15a)$$

Discretizing the z-axis into N levels,

$$\zeta_{diff} \quad = \quad N\,\Delta z\,\hat{k}_z \quad = \quad N\,\Delta z\,\frac{2}{\Delta z}\,\tan\left(\frac{k_z\,\Delta z}{z}\right) \qquad (15b)$$

about
(ζ_{true}
will ei

alar waves,

$$k_z = \sqrt{\frac{\omega^2}{v^2} - k_y^2} \qquad (16)$$

arded scalar waves

$$k_z = \frac{1}{v}\left(\omega - \right. \qquad (17)$$

al loss by making the 15°

ay simplify the algebra with no $\sqrt{\omega^2 - v}$
ximation. Then k_z becomes

$$(18)$$

concept

$$k_z =$$

tizing the y- and t-axes, k_z b $\frac{\hat{k}_y^2}{2\hat{\omega}}$ (19)

k_z = ecomes $\frac{\hat{k}_y^2}{2\hat{\omega}}$

ll occur at the highest fre-
steepest dip k_y/ω the worst e you need to decide on an
y ω. You need to estimate thos o be a half-wavelength or
table phase error. This is often rors w Δz, Δt, Δy to keep
1%. Then you choose eac. Then y any more than this you
$-\zeta_{diff})/\zeta_{diff}$ less than about 1% taken t me, so I stop here.
ther go to sleep or you will disagrh of
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4.4 Absorbing Sid es[1]

Computer memory cells are often used o model points in a volume containing p ies of the finite computer he number of cells is neces- sarily finite, t agating waves. Regrettably, n infinite volume. Waves in the computer ough we often wish to model t the waves had gone away memory a reflect back from the bound puter capacity it is natural to infinity. gh we would usually prefer t nditions. In this section we to try to deve avoid the need for infinite co conditions, not as absorp- will develop so p absorptive side boundary c iously much cheaper. tive as an inf ne highly absorptive boundary tude of appended zeros but ob

Truncation at the Ends of the Cable and at the Ends of the Survey

In explor on we have two kinds of hori ontal truncation problem. The first of se is at the end of the geop hone cable. This problem mainly affec common-midpoint stacking. The second is at the geo- graphical bou aries of the survey. This pro blem mainly affects migra- tion. In each ase the ideal solution is not an infinite amount of zero padding. The ideal solution is some kind of a horizontal extrapolation of the dataset. wo important ingredients to s ch an extrapolation are a noise model, d a wave-propagation signal m odel. Such an ideal extra- polation is ra ly, if ever, approached in prac tice. Usually we settle for zero padding d possibly some tapering. For stacking, the zero padding has no cost. e migration situation seems t be analogous to stacking in that hyperb loids are collapsed to points. ctually the two situations are different ecause of the data itself. h migration we can have reflectors dip g either downward or upward toward the end of the sec- tion. When the energy dips downward, then igration will tend to move the energy bac into the section. This is al the case with stacking in that energy mo es from the far end of the ca e back toward zero offset. It is in the ot r situation, the one which o curs only with migration, which is the tr ublesome one. In this case e ergy will tend to move off the mesh on w ich the data is defined. Again philosophically, the best solution is some kind of an extrapolation of the dataset. In practice, the zero-slope side ondition is perfect when dips vanish, and it is tolerable when they are small. Difficulty occurs when upward dipping energy is seen at the edg s of the survey. On migration it tends to falsely reflect back into the dataset. Such spurious reflections are usually

[1] SEP-25, pp 277-283.

nable.

problem may be reduced by appending zeros to the sides of the , thus providing the dipping energy with a place to go. This is but it does not solve the whole problem for two reasons. The and more basic reason is that some kind of an edge diffraction l be produced. A secondary reason is that the zero padding can- ntinue for an infinite distance. A diminishing value-for-cost s that the zeros terminate somewhere beyond the end of the sur- the termination there is a reasonable chance of finding remain- rgy which we would prefer to absorb rather than reflect. It is this n which Engquist solved. The main practical benefit is the ic one of reduction in the number of padded zeros. To under- ngquist's solution we begin with a different problem, which is con- ly easier.

st Boundaries for the Scalar Wave Equation

simplest boundary condition is that a function should vanish on indary. A wave incident onto such a boundary reflects with a in polarity (so that the incident wave plus the reflected wave will on the boundary). The next-to-simplest boundary condition is the pe boundary condition. It is also a perfect reflector, but the on coefficient is +1 instead of -1. Two points at the edge of the cing mesh are required to represent the zero-slope boundary. st general boundary condition usually considered is a linear com- n of function and slope. This is also a two-point boundary condi- so happens that our extrapolation equations contain only a single erivative so that on the z-axis they are a two-point condition. ng this, Björn Engquist recognized a new application for extrapo- quations. Many researchers in other disciplines are interested in l modeling, that is, evolving forward in time with an equation like alar wave equation, say $P_{xx} + P_{zz} = P_{tt} / v^2$. These people y suffer the consequences of limited memory. Engquist's idea was ey should use our extrapolation equation as their boundary condi- Suppose they desire an infinite absorbing volume surrounding a the (x,z)-plane. Then they need a boundary condition going all around the box. They could use our downgoing wave equation on tom of the box and our upcoming wave equation on the top edge. es could be handled analogously with an interchange of x and z. ea was thoroughly tested and confirmed by Robert Clayton. An e of one of his comparisons is in figure 1.

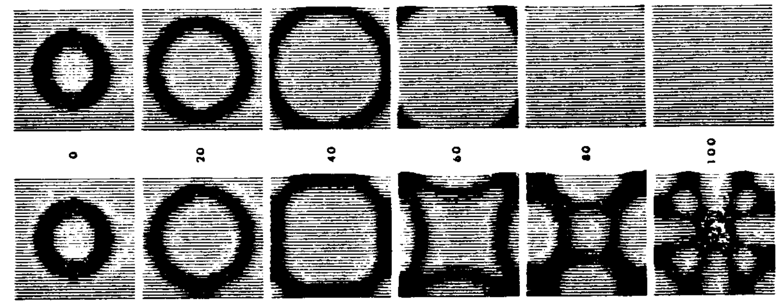

FIG. 1. (Clayton). Expanding circular wavefront in a box with absorbing sides (top) and with zero-slope sides (bottom).

Engquist Side Conditions for the Extrapolation Equations

In data processing we use the extrapolation equation in the interior of the region under study. This is unlike the forward modeling in which the full scalar wave equation is used in the interior and an extrapolation equation can be used on the boundary. The scalar wave equation has a circular dispersion relation whereas the extrapolation equation ideally has a semi-circular one. Reasoning by analogy, Engquist speculated that a quarter-circular dispersion relation might be some sort of ideal side boundary for wave-extrapolation problems. To be more specific and immediately applicable he proposed that the quarter circle be approximated by a straight line. This is depicted in figure 2.

The advantage of the straight-line dispersion relation is that in the space domain it represents a very simple, first-order, differential equation. A first-order equation has first derivatives which can be expressed over just two data points. Thus it can be used as a conventional, two-point, side-boundary condition. The right-side equation on figure 2 defines the boundary dispersion relation D.

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wave of

$$0 = \frac{vk_z}{\omega} - 1 + const\ \frac{k_x}{\omega} = D(\omega, k_x, k_z) \quad (1)$$

z)-space it is

$$0 = (v\ \frac{\partial}{\partial z} + \frac{\partial}{\partial t} + const\ \frac{\partial}{\partial x})\ P \quad (2)$$

ed time, $\partial / \partial z$ may be eliminated with the interior equation.

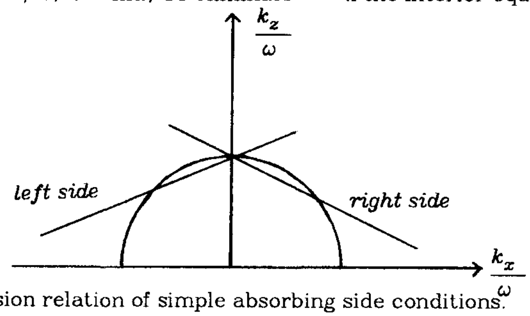

FIG. 2 Dispersion relation of simple absorbing side conditions.

In (... athematical point of view which is non-physical is to imagine culiar physics which prescribe that the physical equation which n some region is just that which has the dispersion relation of rbing side condition. Aside this fictitious region is another in e extrapolation equation applies. At the point of contact the In rel... must match. It may come as no great surprise that the smal- ndary reflections occur where the two dispersion relations are a tch to each other. So the slope of the straight line is picked to ood fit over the range of angles of interest. A nice example of ndary absorption for the diffraction equation is shown in figure is a reproduction of a result of Clayton in SEP-10, p 24.

ne Reflection Coefficient

s look at some of the details of the reflection coefficient calcula- mathematical expression for a unit amplitude monochromatic ve incident on the side boundary superposed with a reflected nagnitude c is given by

$$P(x,z) = e^{-i\omega t + i k_z z} \left(e^{+i k_x x} + c\ e^{-i k_x x} \right) \quad (3)$$

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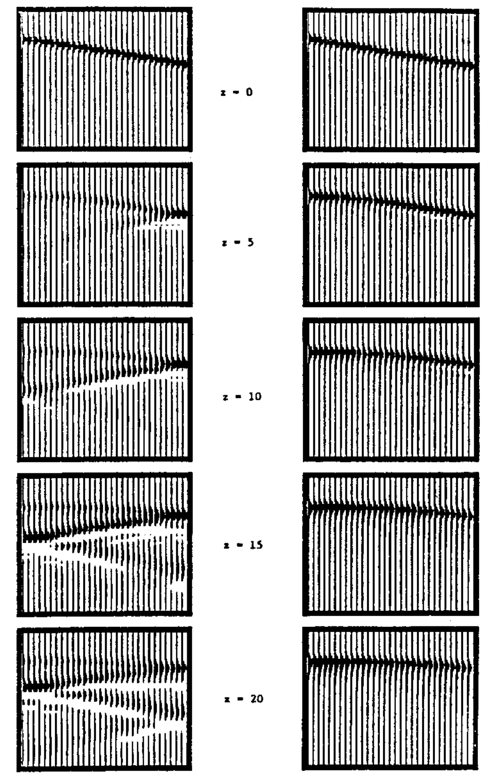


"Reflecting Sides Absorbing Sides"

FIG. 3. (Clayton) A comparison of zero-slope side conditions versus absorbing sides. In this figure it is the diffraction equation, not the migration equation, which is used. The first arriving energy is drifting rightward and being absorbed at the right boundary. No energy enters at the left boundary. So we see a weakening diffraction on the left. On the right the amplitudes appear unchanged because each trace is rescaled to unit amplitude for display.

...ation (3) we understand ω and k_x to be arbitrary and k_z to be
...ined from ω and k_x by the dispers...on relation of the *interior*
i.e. a semi-circle approximation. Ass...ming this interior solution
applicable at the side boundary we may insert (3) into the
...ntial equation (2) which represents th... side boundary. Doing so
... $\partial/\partial x$ converted to $+ik_x$ on the incident wave, $\partial/\partial x$ con-
... to $-ik_x$ on the reflected wave, and $\partial/\partial z$ converted to ik_z.
...e first term in (3) produces the dispersion relation $D(\omega,k_x,k_z)$,
...the amplitude P. The second ter... produces the reflection
...ent c times $D(\omega,-k_x,k_z)$, times P ...

$$c \quad = \quad \frac{-D(\omega,k_x,k_z)}{D(\omega,-k_x,k_z)} \qquad (4)$$

...se of zero reflection arises when th...e numerical value of k_z
...d by the interior equation at (ω,k_x) happens also to satisfy
the dispersion relation D of the side boundary condition. That
...s why we try to match the quarter circle as well as we can. You
...suspect that the straight line dispersion relation corresponds to
...st general form of a side boundary condition, which is expressible
... two end points. Actually a more general expression with an extra
...ble parameter b_3 which fits even better is

$$D(\omega,k_x,k_z) \quad = \quad \left(1 - b_3 \frac{vk_z}{\omega}\right) \frac{vk_z}{\omega} \;-\; \left(b_1 - b_2 \frac{vk_x}{\omega}\right)$$

...oice was considered in more detail by Claerbout and Clayton in
... In fact figure 3 was computed with this side condition.

...y

...erbout and Clayton established abso...
...undaries for the 15° equation includ...

Unfortunately an air-tight analysis of... ...ute stability of absorbing
...e framework of the Muir impedance ... the discretization of the
the stability has been established for ... stability seems to be out-
...te that a stability analysis may event... ...ules. Consequently I don't
cut" analysis in SEP-20, but that is ... the 45° equation. I'd like to
in examples, but that doesn't prove it ... ...ually fit within the "causal
... pure speculation. It has
...always will.

4.5 Stretching Tricks

Stretching tricks enable Fourier analysis to work, at least approximately. I've never really been too interested in this myself, so I'll do a lousy job of presenting it. Since it might be useful in practice, I'll give you the idea, and point to the references.

Gazdag's Method for V(x) .... **not here yet**

Stolt's Stretch .... **not here yet**

olate causality outside that range
al equations
diculous sens s obviously a great need to know
ions which a plicable. Luckily in seismic imag
erest might n stability the main requirement is
and use it. Of all the virtues a com

4.6 Imp

uation there ity, accuracy, clarity, generality, s
tuations is tant seems to be stability.

In describing given to the stabil (2). To regai that since the phys ility analysis sonably accurate n have - stabi with stability, like e most impo as highly academic

zero you take an infinite
ne terms in two different ways, an

Quite the oppo ing where we are ing, the way nature information abou inverse modeling depth. Nature doe problems, so we ca polate downward. cannot determine

one equal) + · · · = 1 + (−1+1) + (−1+1)
. . . , group + · · · = 1 + 0 + 0 + · · ·

$(1-1) + (1-1)$
$+ 0 + 0$

Instability com

1) Mathematic may be a
2) Approxima range of in

In any practical si of the above two s are usually in case we learn some stab tional algorithm ca modularity, etc.

ntion ing is d rea- ncern arded

ocess- Model- acting nts is ion in value extra- imply

Beware of Infinity!

To prove tha 1, −1, +1, −1, +1, them thus:

$(1-1) +$

0

Wave-Field Extrapolation[1]

al processes rarely is much att eling equations. The common fee there stable, so must be any correct an

pedance and ions. This is so often true that co uency stable physic existence proofs, is frequently reg ty of the mod

ical process nce applies in geophysical data pr which nodeling equa he *inverse* of physical modeling. ng we concern with trapolation forward in time. Extra that nterior from surface measureme puta-

site circumsta on is really extrapolating informa speed, nvolved with lue problems in depth, not initia does it, is e rselves lucky when we are able to the earth's extrapolation is unstable then we s Such extrac n we seek.

s boundary v rom one of the following two cause series n consider ou d add When a depth may have a unique solution, but he informatio tivity to data accuracy.

e reasonable and valid in the freq + · · ·

values −1<Z<+1 er way to say it, is to say $U(Z$ define
Z-transform would otice that this range includes $+ x_2$
zero. More general infinite if the coefficients u
y, u_t is causal if $|U(Z)|$ one
volu

delay

proves that we ext let us have terms may be sults. Let a pie ded in two, giv- nto two eighths. 1/16, ... No t back into the

caus nite number of ed if the sum of are called *abso-*

$0 = 1$

d by

$Z^2 + \cdots$

Of course this does not prove that one equals ze must be very care l in dealing with infinite se another infinite seri s where it is perfectly clear t regrouped into any rder without fear of paradox be divided into halve s. Then let one of the halves ing quarters. Then if the two quarters can be div Continue likewise he infinite series is 1/2, 1/4, matter how the pie es are rearranged, they shou pie plate and exact fill it.

The danger of finite series is not they have terms but that they may sum to infinity. Safety is the absolute values of the terms is finite. Such s *lutely convergent.*

ime unit. Then $(Z)U(Z)$ and ion and cross-

operator Z it finite. Numeri- +1, $Z=-1$, and say $|Z|=1$ or able. When you circle. In this sform. We can t of energy by the unit circle

Z-transform

The Z-transform of an arbitrary function x_t is

$$X(Z) = \cdots x_{-2}Z^{-2} + x_{-1}Z^{-1} + x_0 + x_1 Z$$

Give Z the physical interpretation of time shift by Z^2 delays two time units. Expressions $X(Z)\bar{U}(1/Z)$ are useful because they imply correlation of the ... domain coefficients.

al is to say that nite for all real nd at $Z=0$ the *etc.* were not the unit disk

Going on to consider numerical values for the turns out to be useful to ask whether $X(Z)$ is finite cal values of Z which are of particular interest all those complex values of Z which are unit magn $Z = exp(i\omega\Delta t)$ where ω is the real Fourier transfor think of ω being rea you are thinking of Z on t case the Z-transform is called a discrete Fourier restrict attention to time functions with a finite demanding that $U(Z)$ be finite for all values of $|Z|=1$.

The most straight forward way to say that u_t is

$|Z| \le 1$.

Now consider an example from time-series analysis. The expres $1/(1-2Z)$ can be expanded into powers of Z in (at least) two di ways. We have

$$\frac{1}{1-2Z} \quad = \quad 1 + 2Z + 4Z^2 + 8Z^3 + \cdots$$

$$= \quad -\frac{1}{2Z}\,\frac{1}{1-\frac{1}{2Z}} \quad = \quad \frac{-1}{2Z}\left[1 + \frac{1}{2Z} + \frac{1}{4Z^2} + \right.$$

sion
rent

Which of these two infinite series is convergent depends on the nume value of Z. For $|Z|=1$ the first series is divergent, but the second verges. So the only acceptable filter is *anticausal*. Can we say th series expansion is unique? To do so, we must demand that it conve Complex-variable theory goes into this with greater depth.

Now go a little further and define an inverse filter $A(Z)=1/B$ We have seen that whether $A(Z)$ is causal is a question of whether finite *everywhere* inside the unit disk. This is really a questio whether $B(Z)$ vanishes *anywhere* inside the disk. For exa $B(Z)=1-2Z$ vanishes at $Z=1/2$. There $A(Z)=1/B(Z)$ must infinite, that is to say, the series $A(Z)$ must be non-converge $Z=1/2$. So a_t would be non-causal. A most interesting case, *minimum phase* is when both a filter $B(Z)$ and its inverse are ca

ical
con-
at a
ges.
(Z).
it is
n of
iple,
be
it at
lled
isal.

In summary

causal	$\lvert B(Z)\rvert < \infty$ for $\lvert Z\rvert \le 1$
causal inverse	$\lvert 1/B(Z)\rvert < \infty$ for $\lvert Z\rvert \le 1$
minimum phase	*both above conditions*

Review of Impedance Filters

Use Z-transform notation to define a filter $R(Z)$, its input X and output $Y(Z)$. Then

$$Y(Z) \quad = \quad R(Z)\,X(Z)$$

The filter $R(Z)$ is said to be *causal* if the series representation of R has no negative powers of Z. In other words, y_t is determined

(Z),

(Z)
rom

ally, the filter $R(Z)$ will be ive powers of Z. This means it and past values of y_t by

present and past values of x_t. Addition *minimum phase* if $1/R(Z)$ has no negat that x_t can be determined from preser straightforward polynomial division in

phase, additionally it can be work is represented by

$$X(Z) \quad = \quad \frac{Y(Z}{R(Z} \quad = \quad \sum_t \textit{voltage} \times \textit{current}$$

Given that $R(Z)$ is already minimum an impedance function if positive energy o

$$0 \le \textit{work} \quad = \quad \sum_t \textit{force} \times \textit{velocity} \quad \frac{1}{Z}\Big]\, Y(Z) + \bar{Y}\left(\frac{1}{Z}\right) X(Z)\Big]$$

$$= \quad \frac{1}{2}\sum_t \left(\bar{x}_t y_t + \bar{y}_t x_t\right)$$

$$= \quad \textit{coef of } Z^\circ \textit{ of } \left[\bar{X}\right. \qquad = \quad \int \bar{X}\, X\, \mathrm{Re}\,(R)\, d\omega$$

$$= \quad \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\,(\bar{X}\,Y)\, d\omega$$

ocated at any ω, it therefore In summary,

$$= \quad \int \mathrm{Re}\,(\bar{X}\,R\,X)\, d\omega$$

Since $\bar{X}X$ could be an impulse function follows that $\mathrm{Re}\,[R(\omega)] \;\ge\; 0$ for all real ω

Conditions for a function to be an *impedance function:*	
causality	$r_t = 0$ for $t < 0$ $\quad\lvert R(Z)\rvert < \infty$ for $\lvert Z\rvert \le 1$
causal inverse	$\lvert 1/R(Z)\rvert$ $< \infty$ for $\lvert Z\rvert \le 1$
dissipates energy	$\mathrm{Re}\,[R(\omega)] = R(Z$ $) + \bar{R}(1/Z) \ge 0$ *real* ω

Adding an impedance function to its Fourier conjugate we get a purely positive function (imaginary part is zero) like a power spectrum, say

$$\left[r_0 + r_1 Z + r_2 Z^2 + \cdots\right] + \left[\bar{r}_0 + \bar{r}_1\frac{1}{Z} + \bar{r}_2\frac{1}{Z^2} + \cdots\right] \quad \ge \quad 0 \quad \text{for } \textit{real}\ \omega$$

$$R(Z) + \bar{R}\left(\frac{1}{Z}\right) \quad \ge \quad 0 \quad \text{for } \textit{real}\ \omega$$

which is the basis for the statement that "the impedance time function is one side of an autocorrelation function."

Impedances also arise in economic theory where X and Y are price and sales volume. Then I suppose that the positivity of the impedance means that in the game of buying and selling you are bound to lose!

Causal Integration

Begin with a time function p_t. We define its Z-transform by

$$P(Z) \quad = \quad \cdots p_{-2}Z^{-2} + p_{-1}Z^{-1} + p_0 + p_1 Z + p_2 Z^2 + \cdots$$

Define an operator $-i\hat{\omega}\Delta t$ by

$$\frac{1}{-i\hat{\omega}\Delta t} \quad = \quad \frac{1}{2}\,\frac{1+Z}{1-Z}$$

We define another time function q_t with Z-transform $Q(Z)$ by applying the operator to P

$$Q(Z) \quad = \quad \frac{1}{2}\,\frac{1+Z}{1-Z}\,P(Z)$$

Multiply both sides by $(1-Z)$

$$(1-Z)\,Q(Z) \quad = \quad \frac{1}{2}(1+Z)\,P(Z)$$

Equate the coefficient of Z^t on each side

$$q_t - q_{t-1} \quad = \quad \frac{p_t + p_{t-1}}{2}$$

Taking p_t to be an impulse function we see that q_t turns out to be a step function, that is,

$$p \quad = \quad \cdots\ 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, \cdots$$

$$q \quad = \quad \cdots\ 0, 0, 0, 0, 0, \tfrac{1}{2}, 1, 1, 1, 1, 1, 1, \cdots$$

So q_t is the discrete domain representation of the integral of p_t from minus infinity to time t. It is the same as a Crank-Nicolson style integration of the differential equation $dQ/dt = P$. The operator $(1+Z)/(1-Z)$ is called the bilinear transform. The accuracy of the approximation to differentiation is seen to be

$$-i\frac{\hat{\omega}\Delta t}{2} \;=\; \frac{1-Z}{1+Z} \;=\; \frac{Z^{-.5}-Z^{+.5}}{Z^{-.5}+Z^{+.5}} \;=\; -i\,\frac{\sin(\omega\Delta t/2)}{\cos(\omega\Delta t/2)} \;=\; -i\,\tan\!\left(\frac{\omega\Delta t}{2}\right)$$

We may note that the integration operator has a pole at $Z=1$ which is exactly on the unit circle. This raises the possibility of the paradox of infinity. In other words there are other non-causal expansions too. To avoid any ambiguity we introduce a small positive number $\varepsilon = 1-\rho$. Now the integration operator becomes

$$\begin{aligned} I \quad &= \quad \frac{1}{2}\,\frac{1+\rho Z}{1-\rho Z} \\ &= \quad \frac{1}{2}(1+\rho Z)\left[1 + \rho Z + (\rho Z)^2 + (\rho Z)^3 + \cdots\right] \\ &= \quad \frac{1}{2} + \rho Z + (\rho Z)^2 + (\rho Z)^3 + \cdots \end{aligned}$$

Because ρ is slightly less than one this series converges for any value of Z on the unit circle. If we had chosen a small negative ε instead of a positive one we would have found it necessary to make an expansion in negative powers of Z instead of positive powers.

Now the big news is that the causal integration operator is an example of an impedance function. It is clearly causal with a causal inverse. Let us check in the frequency domain that the real part is positive. Rationalizing the denominator we have

$$\begin{aligned} I \quad = \quad \frac{1}{2}\,\frac{(1+\rho Z)}{(1-\rho Z)}\,\frac{(1-\rho/Z)}{(1-\rho/Z)} \quad &= \quad \frac{(1-\rho^2) + \rho(Z - 1/Z)}{\textit{positive}} \\ &= \quad \frac{(1-\rho^2) - 2i\rho\sin\omega\Delta t}{\textit{positive}} \end{aligned}$$

Again it is our choice of a positive ε which has caused $1-\rho^2$, hence the real part to be positive for all ω as shown in figure 1.

As multiplication by $-i\omega$ in the frequency domain is associated with differentiation d/dt in the time domain, so is division by $-i\omega$ associated with integration. People usually associate the asymmetric operator (1,-1) with differentiation. But notice that the inverse to the causal integration operator, namely

$$I^{-1} \quad = \quad 2\,\frac{1-\rho Z}{1+\rho Z}$$

$$I^{-1} \quad = \quad 2 - 4\rho Z + 4(\rho Z)^2 - 4(\rho Z)^3 + \cdots$$

also represents differentiation, but it is completely causal and not at all asymmetric. In fact, in linear systems analysis this is often the preferred discrete representation of differentiation. As we will see, the construction of higher-order stable differential equations must now be and zero frequency are on the left end of their respective axes.

subject to the rules which we will develop for combining impedance functions.

Occasionally it will be necessary to have a *negative* real part for the differentiation operator. This can be achieved by taking ε negative which means taking $\rho > 1$ and doing the infinite series expansion in powers of Z^{-1}, that is, anticausally instead of causally. In either case the imaginary part will still be $-i\omega$ but the real part has opposite sign.

Functional Analysis

We will establish, in sequence, the following theorems about exponentials, logarithms and powers of Fourier transforms of filters:

1. The exponential of a causal filter is causal.
2. The exponential of a causal filter is a minimum phase filter.
3. The phase curve of a minimum phase filter does not enclose the origin.
4. The logarithm of a minimum phase filter is causal.
5. Any power of a minimum phase filter is minimum phase.
6. Any real fractional power $-1 \le \rho \le 1$ of an impedance function is an impedance function.

To establish Theorem 1 we define the Z-transform of an arbitrary causal function

$$U(Z) \quad = \quad u_0 + u_1 Z + u_2 Z^2 + \cdots \qquad (1)$$

and substitute it into the familiar power series for exponential

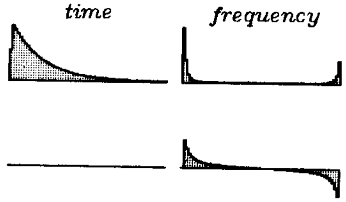


FIG. 1. The causal integration operator I. The frequency axis is represented by discrete Fourier transform over 256 points. Zero time and zero frequency are on the left end of their respective axes.

$B_- \;=\; e$

Clearly both B_+ and B_- are cause another. A minimum phase filter is de inverse. So B_+ and B_- are minimum

Theorem 3 refers to the phase ang minimum phase filter. The phase angle of the ratio $[\mathrm{Im}\, B(Z)]/[\mathrm{Re}\, B(Z)]$. As arctangent must return to the same val augmented by 2π if the phase curve e need not be a periodic function of ω minimum phase function $U(Z) = Z = e$ not a periodic function of ω even thoug tion of ω. For a minimum phase wavelet

$$\frac{U^3}{3!} + \cdots \qquad (|U| < \infty) \qquad (2)$$

will be generated so that $B(Z)$ the factorials in the denomina- so $B(Z)$ is always causal. exponential is not just causal but

$$B \;=\; e^{U(Z)} \;=\; \exp\left(\sum_{k=0}^{N} U_k \cos \ldots \right. \qquad (3a)$$

$$= \; \exp[r(\omega) + i\,\varphi \ldots \qquad (3b)$$

$+U$ … $-U$ … 1 and they are inverses of one

The phase φ being a sum of periodic fu tion of ω, which means that in the p representing $B(\omega)$ does not enclose the

fined to be causal with a causal phase.

Now we set out to establish the Theorem 4, that the logarithm of a mini the logarithm of (2) and form the Z-deriv

le in the Fourier domain of the φ is defined by the arctangent ω ranges from 0 to 2π, the ie. But the phase itself would be ncloses the origin. So the phase For example, the causal, non- has a phase equal ω which is h its tangent is a periodic func- we have

$$U \;=\; \ln \ldots$$

$$\frac{dU}{dZ} \;=\; u_1 + 2u_2 Z + 3u_3 Z^2 + \cdots \qquad (4b)$$

$$\frac{dU}{dZ} \;=\; \frac{1}{B} \frac{dB}{dZ} \qquad (4c)$$

$$\ldots s\,k\omega + i \sum_{k=0}^{N} U_k \sin k\omega \Big)$$

$(\omega)]$

nctions, is itself a periodic function ane of $(\mathrm{Re}B, \mathrm{Im}B)$ the curve origin.

onverse of Theorem 2, namely num phase filter is causal. Take ative

(4a)

$$B(Z) \;=\; e^{U} \;=\; 1 + U + \frac{U^2}{2!} \ldots$$

It is clear that no negative powers of Z will have no negative powers of Z. Als tor assure us that (2) always converges,

To establish Theorem 2, that the e also minimum phase, we consider

$B_+ \;=\; e$

we assume B is minimum phase it means that both $1/B$ and dZ on the right of (4c) are causal. Since the product of two causals ...usal, we have dU/dZ causal. But clearly dU/dZ could not be ...l unless U is causal. That proves it except for the remote danger B might converge while dB/dZ diverges.

...n to Theorem 5, which says that any power of a minimum-phase ...ion is minimum phase. Consider

$$B^r \;=\; \left(e^{\ln B}\right)^r \;=\; e^{r \ln B} \tag{5}$$

B is assumed minimum phase, $\ln B$ by Theorem 4 will be causal. ...ing by a constant r does not change causality. Exponentiating ..., by Theorem 2, that B^r is minimum phase.

...nally we will prove Theorem 6, that an impedance function can be ...d to any real fractional power $-1 \le \rho \le +1$ and the result is still an ...dance function. An impedance function is defined as a minimum- ... function with the additional property that the real part of its ...er transform is positive. This means that the phase angle φ lies in ...nge $-\pi/2 < \varphi < +\pi/2$. Raising the impedance function to the ρ ... will compress the range to $-\pi\rho/2 < \varphi < \pi\rho/2$. This will keep its ...art positive. Theorem 5 states that *any* power of a minimum-phase ...on is causal, which is a lot more than we need to be certain that a ...*onal* power of an impedance function will be causal.

Rules for Compounding Impedance Functions

...e of the difficulties in applied geophysics is this: Results may have ...al utility only in a certain limited range of frequencies, and rea-...le approximations may be made in that range. But if a spectrum ...edance becomes negative outside the applicable range, say near ...yquist folding frequency, then the calculation (by Murphy's Law) ... unstable and hence useless. Thus Muir's rules[2] for compounding ...ance functions deserve careful attention. Let R' denote a new ...ance function generated from old known impedance functions R_1, R_2.

...uir's rules are:

i1: Multiplication by positive scalar a $\qquad R' = a\,R$

i2: Inversion $\qquad R' = \dfrac{1}{R}$

i3: Addition $\qquad R' = R_1 + R_2$

Proofs:

i1: Obviously preserves causality and positivity of real parts of F.T.

i2: Causality is OK since by definition every impedance is minimum phase. Positivity follows since for any ω we have $1/(a+ib) = (a-ib)/(a^2+b^2)$.

i3: Causality and positivity are trivial. Proof that the inverse of $R_1 + R_2$ will be causal is a bit harder. Since the real part of $R_1 + R_2$ is positive, its phase doesn't loop around the origin. Thus it must be minimum phase so its inverse is causal.

An abstract though excellent proof of *i3* is found in complex variable theory. The analyticity of R_1 and R_2 inside the circle implies that the sum is analytic there. Positivity of the real part on the circle and LaPlace's equation inside implies that $\mathrm{Re}(R_1 + R_2)$ does not vanish inside, so $|1/(R_1 + R_2)| < \infty$ inside $|Z| \le 1$, which suffices.

Isomorphism with Reflectance Function C

Given any impedance function R, then the following equation defines an associated reflectance function C

$$C \;=\; \frac{1-R}{1+R} \tag{6a}$$

We will see that the reflectance function is also causal and that it is less than unity in magnitude, say

$$|C|^2 \;=\; \bar{C}\left(\frac{1}{Z}\right) C(Z) \;<\; 1$$

Causality follows because the numerator $1 - R$ is causal and the denominator, being the sum of two impedance functions, has a causal inverse. The product of two causals is causal. That the magnitude of C is less than unity follows from noting that the magnitude of the numerator is less than the magnitude of R and the magnitude of the denominator is greater. Unlike the impedance function $R(Z)$, the reflectance function $C(Z)$ is not necessarily minimum phase. An example is $R = 1 + Z/2$, $C = -.5Z/(2 + Z/2)$.

Equation (6a) may be solved for R:

$$R \;=\; \frac{1-C}{1+C} \tag{6b}$$

[2] ...al communication with Francis Muir.

'=*causal* and $|C| < 1$ alone will ensure that R
We may now inquire if ... is an impedance functio... and bottom by $1 + \bar{C}$:
Multiply (6b) on top ...

$$R = \frac{(1 - C)(1 + \bar{C})}{|1 + C|^2}$$

$$= \frac{(1 - \bar{C}C) + (-C + \bar{C})}{positive}$$

$$= (real) + (imag)$$

...he real part is ensured by $|C| < 1$. The causal-
Clearly the positivity of ...erator of (6b) is assumed causal and the denom-
ity follows since the nu...sitive real part (since $1 > |C|$). In summary,
inator is causal with p...eliably produce an impedance function from any
then, equation (6b) will r...tion.
apparent reflectivity func...

...e Extrapolation

Example: Wide-Angle Wa... the causal positive discrete representation of
Let $s = -i\hat{\omega}$ deno...or, say
the differentiation operat...

$$s = -i\hat{\omega}\Delta t = 2\,\frac{1 - \rho Z}{1 + \rho Z}$$

...ursion starting from $S_0 = s$:
Consider the following rec...

$$S_{n+1} = s + \frac{X^2}{s + S_n}$$

...this recursion as a means of developing wide-
Francis Muir introduced ...imations for migration and he developed his angle square-root appro...to show that every S_n is an impedance func-
three rules i1, i2, and i3 ...ks, first note that the denominator $s + S_n$ is, tion. To see why this wo... impedance functions. Then its inverse is an for n=0, the sum of tw...multiplication by the real positive constant X^2 impedance function, and ... s all preserve the properties of impedance and addition of another ...es large this recursion either converges or it functions. As N becom...it does, we can see to what it converges by set-
does not. Supposing that ... We have
ting $S_{n+1} = S_n = S_\infty = S$.

$$S = s + \frac{X^2}{s + S}$$

$$S(s + S) = s(s + S) + X^2$$

$$S^2 = s^2 + X^2$$

$$S = \sqrt{s^2 + X^2}$$

$$S = s\sqrt{1 + \frac{X^2}{s^2}}$$

In wave extrapolation problems X^2 is $v^2 k_x^2$ where v is th... city and k_x is horizontal spatial frequency, namely, the Fou... the horizontal x-axis. The quantities S_n are $i\,k_z$ where ...e wave velo-
Fourier dual to the depth z-axis. The cases n = 0, 1, and ...rier dual to monly referred to as the 5°, 15°, and 45° equations, respe... k_z is the desirability of S being positive real is related to the fac2 are com-
acceptable for $\exp(i\,k_z z)$ to decay with z (when k_z is co...tively. The growth is almost certainly not acceptable. ...t that it is ...mplex), but

Exact Square Root

The general form for stable extrapolation problems seems ... to be

$$\frac{dP}{dz} = -R\,P \tag{7}$$

where convergence is assured by the positive real part of the ... impedance function R. In reflection seismology there is great inte... ...est in the square-root extrapolation operator

$$R = -i\,k_z = \frac{-i\omega}{v}\left(1 - \frac{v^2 k_x^2}{\omega^2}\right)^{1/2} \tag{8}$$

At the moment we are disinterested in the space- or frequency-
dependence of velocity, so we set v=1, obtaining

$$R = \sqrt{(-i\omega)^2 + k^2} \tag{9}$$

In (9) we would like a causal representation of the differentia... ...tion opera-
tor such as

$$-i\hat{\omega} = \frac{2}{\Delta t}\,\frac{1 - \rho Z}{1 + \rho Z} \qquad -1 \ll \rho < 1 \qquad \text{and } Z = e^{i\omega\Delta t} \tag{10}$$

We intend to establish that the following operator is an imped...ance func-
tion

$$R = \sqrt{(-i\hat{\omega})^2 + k^2} \tag{11}$$

First note that $(-i\hat{\omega})$ is causal by (10), which means that $(-i\hat{\omega})^2$ is also causal. Also, k^2 is a delta function at the time origin. Thus R^2 given by (11) is causal. Next, let us look at the phase. Figure 2 shows how the phase of (11) is constructed from its constituents.

Now we have seen that R^2 is causal and from the figure we see that it has a "branch cut" property. That is, the phase of R has the positive real property. Theorem 5 forces R to be causal and minimum phase. That, with the phase defined by figure 1, proves that R, given by (9), is an impedance function. (Muir's proof did not extend to the evanescent region of the square root where the continued fraction expansion is not convergent.)

Fractional Integration and Constant Q

By equation (6) and Theorem 6 we know that fractional powers of integration and differentiation are also impedance functions. In fact, Kjartansson (1979) has advocated the fractional power as a stress-strain law for rocks. The conventional rock-mechanics studies begin with a stress-strain law such as

$$stress \;=\; stiffness \times strain \;+\; viscosity \times strain\text{-}rate$$

which in the transform domain is

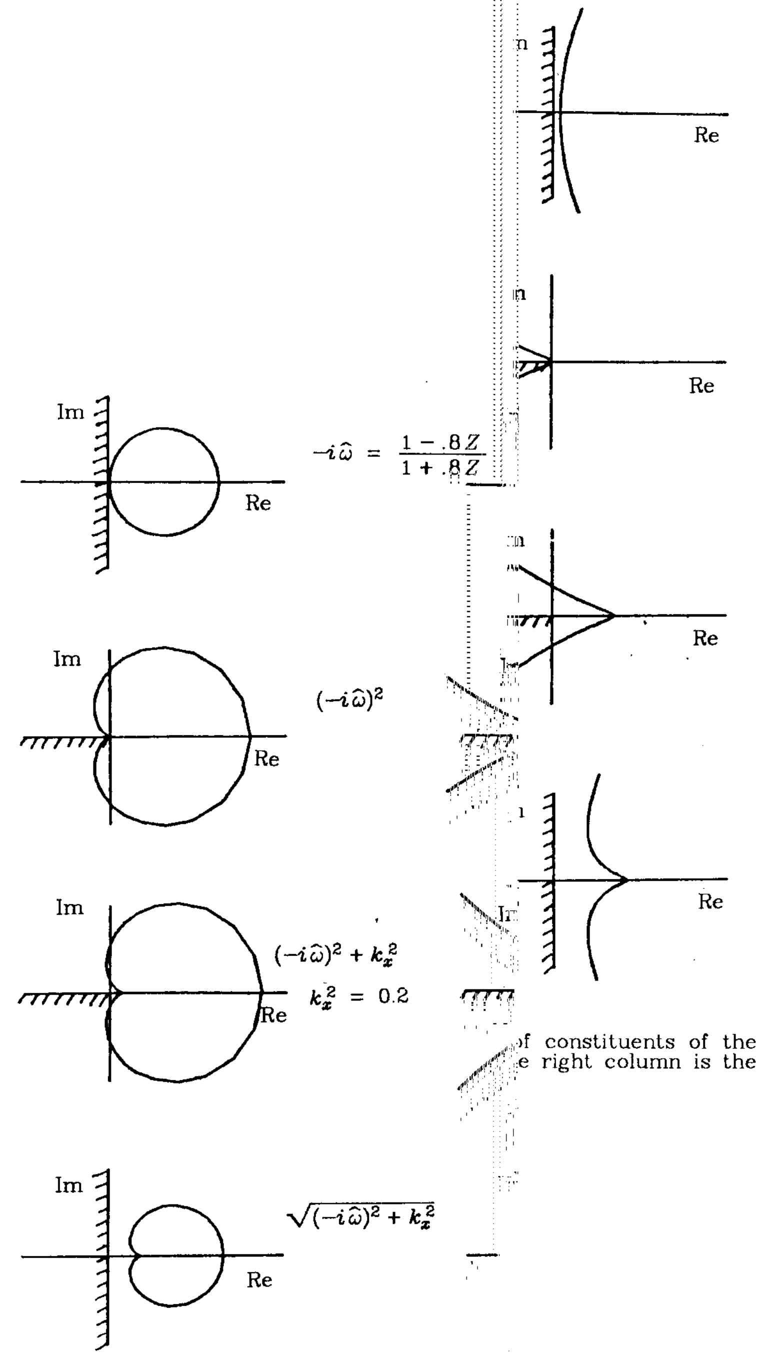
$$stress \;=\; [(-i\omega)^0 \times stiffness \;+\; (-i\omega)^1 \times viscosity]\; strain \qquad (12)$$

Without for the moment considering the physics of the matter, we can consider replacing the arithmetic average of the two terms by a geometric average, say

$$stress \;=\; const \times (-i\omega)^{\varepsilon}\; strain \qquad (13a)$$

$$\;=\; const \times (-i\omega)^{\varepsilon-1}\; strain\ rate \qquad (13b)$$

where ε close to zero gives elastic behavior and ε close to one gives viscous behavior. The fact that $(-i\omega)^{\varepsilon-1}$ is an impedance function meshes nicely with the concepts that (1) stress may be determined from strain history and strain may be determined from stress history, and (2) stress times strain-rate is dissipated power. Kjartansson (1979) points out that $(-i\omega)^{\gamma}$ exhibits the mathematical property called *constant Q*, so that as a stress/strain law for fitting experimental data on rocks, it is far superior to the arithmetic average. To see the constant Q property more clearly, let us express $(-i\omega)^{\gamma}$, in real and imaginary parts:



...f constituents of the ...e right column is the

FIG. 2. (Kjartansson) Complex plane diagram... extrapolation operator R as given by (11). ... same as the left column blown up five times.

$$(-i\omega)^\gamma = |\omega|^\gamma [e^{-i\pi\, sgn(\omega)/2}]^\gamma$$

$$= |\omega|^\gamma \left\{ \cos\left[\frac{\pi\gamma}{2} sgn(\omega)\right] - i \sin\left[\frac{\pi\gamma}{2} sgn(\omega)\right]\right\}$$

$$= |\omega|^\gamma \left[\cos\left(\frac{\pi\gamma}{2}\right) - i\, sgn(\omega) \sin\left(\frac{\pi\gamma}{2}\right)\right] \qquad (14)$$

The constant Q property follows from the constant ratio between the real and and imaginary parts of this function. This is shown in figure 3.

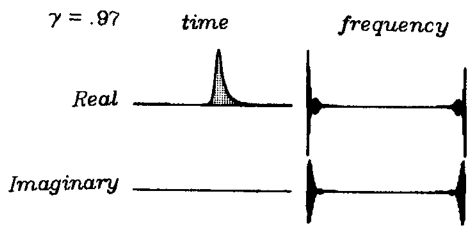


FIG. 3. The constant Q pulse given by $e^{-(-i\omega)^{.97} t_0}$. The frequency axis is represented by discrete Fourier transform over 256 points. Zero time and zero frequency are on the left end of their respective axes.

REFERENCE

Kjartansson, E., 1979, *Constant Q - Wave Propagation and Attenuation*: J. Geophys. Res., v. 84, p. 4737-4748.

EXERCISES

1. Take $\varepsilon < 0$ and expand the integration operator for negative powers of Z. Explain the sign difference.

[illegible]t only that any impedance [illegible]ance C_1, C_2, C', but also that [illegible]to three rules for combining [illegible]s no[illegible] [illegible]eflec[illegible] [illegible]ed [illegible]

2. The word "isomorphism" [illegible] R_1, R_2, R' can be mapped [illegible]out to be one of the three rules Muir's three rules will be mean[illegible] that it is a consequence of the reflectances. [illegible]into a[illegible] independent rule which can be
   a. What are these three [illegible] mapp[illegible] turn[illegible] impedances to make a fourth [illegible]now[illegible]
   b. Although $C' = C_1 C_2$ d[illegible] is a[illegible]ion for square root in an extra- it is obviously true. [illegible]les? [illegible]of th[illegible] three rules or concludes not [illegible] mapped back into the other st[illegible]xpans[illegible] rule. that it [illegible]

3. Consider the fourth-order [illegible]domain [illegible] polation equation [illegible]for the complex frequency

$$\frac{dP}{dz} = i\omega \left[\text{[illegible]} - \frac{1}{2}\left(\frac{vk}{\omega}\right)^2 - \frac{1}{8}\left(\frac{vk}{\omega}\right)^4 \right] P$$

[illegible]aylor e[illegible] [illegible]-domain calculations with the [illegible]ole[illegible]

   a. Will this equation [illegible] [illegible]epend on frequency ω and on $-i\omega = -i\omega_0 + \varepsilon$? Why? [illegible] ti[illegible]e? [illegible]ppose that luckily the velocity
   b. Consider causal and a[illegible]e st[illegible] $(x,\omega) = v_1(x)\, v_2(\omega)$. Obtain a equation. Which, if any, [illegible]may c[illegible] Hints: Try [illegible]ll. S[illegible]

4. Consider material velocity [illegible]ticause [illegible]m [illegible] the horizontal x-coordinat[illegible] is st[illegible]ion. can be expressed in fact[illegible] which [illegible] stable 45° wave-extrapola[illegible] as we[illegible] $- \frac{i\omega}{v_2} f\ (v_1\partial_x)(v_1\partial_x)^T$ [illegible]red fo[illegible]d to the rules in this paper? If [illegible]n equa[illegible]*lue* [illegible]ued fractions.

$$X^2 = positive\ s - \text{[illegible]}$$

[illegible]elate causal integration operator, [illegible]ontin[illegible]e discrete Hilbert transform.

5. Is the Levinson Recursion [illegible] *eigenv*[illegible] boundary between two media so, how? Hint: See Wall's b[illegible]crete[illegible]erent Q is one side of the Hil-

6. Show that the log of [illegible] FGDP [illegible]of th[illegible] $\log[(1+Z)/(1-Z)]$, is o[illegible]ok on [illegible]om a[illegible] [illegible]y di[illegible] Show that the reflected [illegible]the disc[illegible] with the same velocity [illegible]e side [illegible] bert transform. [illegible]ulse f[illegible] [illegible]t sligh[illegible]

Stable extrapolation can be assured by preserving certain symmetries. Given the differential equation

$$\frac{dq}{dz} \;=\; -\mathbf{R}\,\mathbf{q} \qquad (1)$$

and its Crank-Nicolson approximation

$$\frac{\mathbf{q}_{n+1} - \mathbf{q}_n}{\Delta z} \;=\; -\frac{\mathbf{R}}{2}(\mathbf{q}_{n+1} + \mathbf{q}_n) \qquad (2)$$

it will be shown that stability is assured in both cases, provided that $\mathbf{R} + \mathbf{R}^*$ is a positive definite (actually, semi-definite) matrix. In the previous section this subject was developed with the operator $\mathbf{R}$ being a scalar Z-transform. There the matter of *causality* receives much attention, whereas here we focus more on the *matrix* character of $\mathbf{R}$. The present section, along with the previous section provides a theory for causal matrix extrapolation that is useful in time-domain migration. Our purpose for this theoretical work is to enable us to write a "bullet-proof" program for migrating seismic data in the presence of lateral velocity variation. As a final example we will see that the familiar 45° extrapolation equation can be put in the required form.

Stability of the Differential Equation

Let $\mathbf{q}^*$ denote the Hermitian conjugate of $\mathbf{q}$. For equation (1) to be stable, we require the energy $\mathbf{q}^*\mathbf{q}$ to be either constant or decaying as we extrapolate in depth z:

$$\frac{d}{dz}(\mathbf{q}^*\mathbf{q}) \;\le\; 0$$

$$\mathbf{q}^*\mathbf{q}_z + \mathbf{q}_z^*\mathbf{q} \;\le\; 0 \qquad (3)$$

Substituting equation (1) into (3) gives

$$\mathbf{q}^*\mathbf{R}\mathbf{q} + \mathbf{q}^*\mathbf{R}^*\mathbf{q} \;\ge\; 0$$

$$\mathbf{q}^*(\mathbf{R} + \mathbf{R}^*)\mathbf{q} \;\ge\; 0 \qquad (4)$$

Equation (4) shows that $\mathbf{R} + \mathbf{R}^*$ must be positive semi-definite for the differential equation to be stable, which is what we set out to prove.

[1] SEP-16, p 83-87.

4.7 The Bullet Proofing of Muir and Godfrey[1]

…y of the Difference Equation

…stability of the difference equation is shown in a similar way, but …s some extra clutter. Incorporating some ideas contained in a …rom Bjorn Engquist, we reduced the clutter to what is found … First observe the identity:

$$(\mathbf{a}^*\mathbf{a} - \mathbf{b}^*\mathbf{b}) \;\equiv\; \frac{1}{2}[(\mathbf{a}+\mathbf{b})^*(\mathbf{a}-\mathbf{b}) + (\mathbf{a}-\mathbf{b})^*(\mathbf{a}+\mathbf{b})] \qquad (5)$$

… $\mathbf{a} = \mathbf{q}_{n+1}$ and $\mathbf{b} = \mathbf{q}_n$, equation (5) becomes

$$(\mathbf{q}_{n+1}^*\mathbf{q}_{n+1} - \mathbf{q}_n^*\mathbf{q}_n) \;=\;$$

$$\frac{1}{2}[(\mathbf{q}_{n+1}+\mathbf{q}_n)^*(\mathbf{q}_{n+1}-\mathbf{q}_n) + (\mathbf{q}_{n+1}-\mathbf{q}_n)^*(\mathbf{q}_{n+1}+\mathbf{q}_n)] \qquad (6)$$

…place the $(\mathbf{q}_{n+1} - \mathbf{q}_n)$ terms by equation (2):

$$-\frac{\Delta z}{4}[(\mathbf{q}_{n+1}+\mathbf{q}_n)^*\mathbf{R}(\mathbf{q}_{n+1}+\mathbf{q}_n) + (\mathbf{q}_{n+1}+\mathbf{q}_n)^*\mathbf{R}^*(\mathbf{q}_{n+1}+\mathbf{q}_n)]$$

$$=\; -\frac{\Delta z}{4}[(\mathbf{q}_{n+1}+\mathbf{q}_n)^*(\mathbf{R}+\mathbf{R}^*)(\mathbf{q}_{n+1}+\mathbf{q}_n)] \qquad (7)$$

…uation establishes the final result: If the $\mathbf{R} + \mathbf{R}^*$ is a positive … matrix, then $\mathbf{q}_{n+1}^*\mathbf{q}_{n+1}$ is less than $\mathbf{q}_n^*\mathbf{q}_n$.

…ion to 45-Degree Wave-Field Extrapolation

…scalar wave equation for extrapolation of a downgoing wave field

$$\frac{d\mathbf{q}}{dz} \;=\; i\,k_z\,\mathbf{q} \;=\; -\mathbf{R}\mathbf{q} \qquad (8)$$

…the $\mathbf{R}$ operator takes the usual form

$$\mathbf{R} \;=\; -i\,k_z \;=\; \frac{-i\omega}{v}\sqrt{1 - \frac{v^2k_x^2}{\omega^2}} \qquad (9)$$

…plan now is to approximate the square root by the usual series …n and then identify $i\,k_x$ with ∂_x to obtain a space-domain …. The main effort stems from our refusal to make the usual …ion that $v(x,z)$ is independent of x. Since $\partial_x v q$ differs from …e space representation does not seem to be unique, and we may …how the variable q relates to physical wave variables like pres…d displacement. Since (9) is purely imaginary, we will have

efinite. More details may be found in the original rese
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vision sign in (12) is justifiable because the matrix **T**
he identity matrix **I**. (A hazard in this kind of effort is
ommute with the diagonal matrix **V**.) The matrix **M** has
equired of **R** since a basic matrix theorem says that
of a polynomial of a real symmetric matrix are the poly
genvalues. In other words, replacing **T** in (12) by one
produces a complex **M** whose real part is positive so
sitive as required. What we need to show is that the fo
positive definite:

$$\mathbf{R} + \mathbf{R}^* \;=\; \mathbf{V}^{-1/2}(\mathbf{M} + \mathbf{M}^*)\mathbf{V}^{-1/2}$$

is positive definite if for arbitrary x, the scalar $x^* \mathbf{A}$
e diagonal matrix $\mathbf{V}^{-1/2}$ can certainly be absorbed in
till be arbitrary, so the proof is complete.

amming it is a nuisance to put $\mathbf{V}^{-1/2}$ on each side o
Actually you can put $\mathbf{V}^{-1}$ on either side. In general,
atic form such as $\mathbf{q}^* \mathbf{U} \mathbf{q}$ where $\mathbf{U}$ is strictly positive de
asing if

$$\mathbf{R}^* \mathbf{U} + \mathbf{U} \mathbf{R}$$