1.2 Wave Extrapolation As a 2-D Filter[1]

In Fourier analysis we are familiar with the idea that an impulse function (delta function) can be constructed by superposition of sinusoids (or complex exponentials). In the study of time series this construction is used for the *impulse response* of a filter. In the study of functions of space, it is used to make a physical point source.

Taking time and space together, Fourier components can be interpreted as monochromatic waves. Physical optics (and with it reflection seismology) becomes an extension to filter theory. In this section we learn the mathematical form, in Fourier space, of the Huygen's secondary source. It is a two-dimensional (2-D) filter for spatial extrapolation of wave fields.

Rays and Fronts

Figure 1 depicts a ray moving down into the earth at an angle ϑ from the vertical. Perpendicular to the ray is a wavefront. By elementary geometry the angle between the wavefront and the earth's surface is also ϑ. The ray increases its length at a speed v. The speed which is observable on the earth's surface is the intercept of the wavefront with the earth's surface. This speed, namely $v/\sin\vartheta$, is faster than v. Likewise, the speed of the intercept of the wavefront and the vertical axis is $v/\cos\vartheta$. A mathematical expression for a straight line, like that shown to be the wavefront in figure 1 is

$$z \;=\; z_0 - x \tan\vartheta \tag{1}$$

In this expression z_0 is the intercept between the wavefront and the vertical axis. To make the intercept move downward, we replace it by the appropriate velocity times time

$$z \;=\; v\,\frac{t}{\cos\vartheta} - x \tan\vartheta \tag{2}$$

Solving for time we get

$$t(x,z) \;=\; \frac{z}{v}\cos\vartheta + \frac{x}{v}\sin\vartheta \tag{3}$$

Equation (3) tells us what time the wavefront will pass any particular location (x,z). The expression for an arbitrary shifted waveform is $f(t - t_0)$. Using (3) to define the time shift t_0 we have an expression

[1] Adapted from SEP-25, pp 203-208.

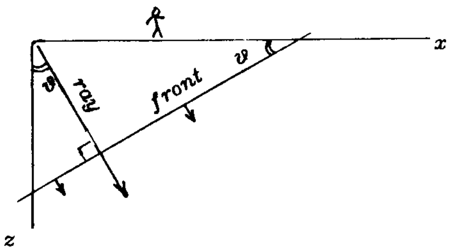


FIG. 1. Downgoing ray and wavefront.

for a wave field which is some waveform moving on a ray.

$$\textit{moving wave field} \ = \ f\left[t \ - \ \frac{x}{v}\sin\vartheta \ - \ \frac{z}{v}\cos\vartheta \right] \qquad (4)$$

Waves in Fourier Space

Arbitrary functions can be made from the superposition of sinusoids. Sinusoids and complex exponentials commonly occur. One reason they occur is because they are the solutions to linear partial differential equations (PDE's) with constant coefficients. The PDE's arise because most laws of physics are expressible as PDE's.

Specializing the arbitrary function in equation (4) to be a cosine function of negative argument times frequency ω, we have

$$\textit{cosine on a ray} \ = \ \cos\left[\omega \left(\frac{x}{v}\sin\vartheta + \frac{z}{v}\cos\vartheta - t \right) \right] \qquad (5)$$

Using Fourier integrals on time functions we encounter the *Fourier kernel* $\exp(-i\omega t)$. To use Fourier integrals on the space-axis x we need to define the spatial angular frequency. Since we will ultimately encounter quite a few different space axes (three for shot, three for geophone, also the midpoint and offset), we will adopt the convention of using a subscript on the letter k to denote the axis being Fourier transformed. So k_x is the angular spatial frequency on the x-axis and $\exp(ik_x x)$ is its Fourier kernel. For each axis and Fourier kernel there is the question of the choice of the sign of i. The sign choice is discussed later in more detail, but essentially we will choose the sign convention of most physics books, namely, to agree with equation (5), which is a wave moving in the positive direction along the space axes. Thus the Fourier kernel for (x,z,t)-space will be taken to be

$$\textit{Fourier kernel} \ = \ e^{ik_x x} \, e^{ik_z z} \, e^{-i\omega t} \ = \ \exp[i(k_x x + k_z z - \omega t)] \qquad (6)$$

Now for the whistles, bells, and trumpets. Comparing (5) and (6) we learn how to relate physical angles to velocity and Fourier components. These relations should be memorized!

Angles and Fourier Components	
$\sin\vartheta = \dfrac{v\,k_x}{\omega}$	$\cos\vartheta = \dfrac{v\,k_z}{\omega}$

(7)

Equally important is what comes next. We may insert the angle definitions into the familiar relation $\sin^2\vartheta + \cos^2\vartheta = 1$. This gives a most important relationship, known as the ***dispersion relation of the scalar wave equation***.

$$k_x^2 + k_z^2 \ = \ \frac{\omega^2}{v^2} \qquad (8)$$

We'll encounter ***dispersion relations*** and the ***scalar wave equation*** later. The reason why (8) is so important is that it enables us to make the distinction between an arbitrary function and an apparently chaotic function which actually is a wave field. Take any function $p(t,x,z)$. Fourier transform it to $P(\omega,k_x,k_z)$. Look in the (ω,k_x,k_z)-volume for any nonvanishing values of P. You will have a wave field if and only if all nonvanishing P have coordinates which satisfy (8). Even better, in practice we often know the (x,t)-dependence at $z=0$, but we do not know the z-dependence. Then we find the z-dependence by the assumption that we have a wavefield, so the z-dependence is inferred from (8).

Migration Improves Horizontal Resolution

In principle, migration converts hyperbolas to points. In practice, we don't get a point. We get a *focus*. A focus has measurable dimensions. Migration is said to be "good" because it increases spatial resolution. It squeezes a large hyperbola down to a tiny focus. To quantitatively describe the improvement of migration, we need to discuss the size of the hyperbola and the size of the focus. Figure 2 shows various ways of measuring the size of a hyperbola.

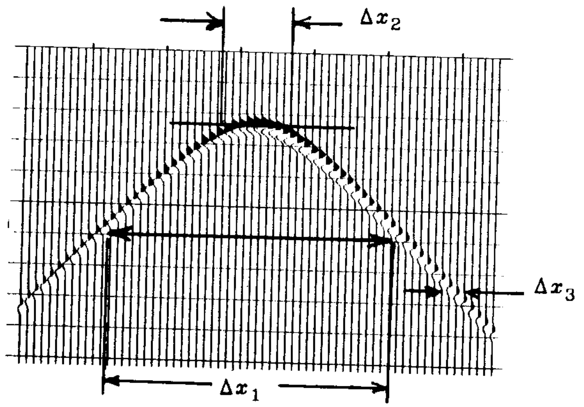


FIG. 2. (Gonzalez) Measurements of width parameters of a hyperbola.

Since the hyperbola is a rather impulsive arrival, we can say that the ω-bandwidth of the hyperbola is approximately given by the zero crossings on the time axis of the main energy burst. I'll mention 50 Hz as a typical value, though you could encounter values four times higher or four times lower. Knowledge of a seismic velocity enables conversion to a depth resolving power. I'll mention 3 km/sec though you could encounter velocities four times greater and four times less. Halving the velocity, to account for two way travel time, we arrive at a depth wavelength of v/f = 30 meters. Whether seismic resolution should be taken as a half wavelength (15 meters) or a smaller value is an issue which involves signal-to-noise considerations outside our present study.

Now for the lateral resolution. First we need to measure the "width" of the hyperbola and then the width of the focus. Figure 2 shows three widths. The widest, Δx_1, includes about three quarters of the energy in the hyperbola. Next is the width Δx_2, called the *Fresnel Zone*. It is measured across the hyperbola at the time when the first arrival has just changed polarity. Third is the smallest measurable width, found far out on a flank. This width, Δx_3, is the shortest horizontal wavelength to be found. Because the hyperbola is an impulsive arrival, we can take this value, Δx_3 to be indicative of the bandwidth of the spatial k_x spectrum. How small a focus can migration make? It will be limited by the available bandwidth in the k_x spectrum. We may simply conclude that the size of the focus will be about the same as Δx_3. (Resolution is the study of the size of error, and it is not awfully useful to be precise about the error in the error.)

What is the meaning of the Fresnel width Δx_2 ? Suppose you and a friend are on opposite sides of a wall (Berlin, maybe). You are both some distance from the wall and begin shouting to each other through a large hole. How does the loudness of the sound depend on the size of the hole ΔX ? It is not obvious, but it is well known, both theoretically, and experimentally, that holes larger than the Fresnel zone cause little attenuation, but smaller holes restrict the sound in proportion to their size.

Wave propagation is a convolutional filter which smears information from a region Δx_2 along a reflector (or Δx_1 in the subsurface) to a point on the surface. Migration, the reverse of wave propagation, is the deconvolution operation. The final amount of lateral resolution is limited by the spatial bandwidth of the data.

A basic fact of seismology is the resolution limitation caused the increase with depth of the seismic velocity. What happens is that as the waves get deeper into the earth, their wavelengths get longer because of the increasing velocity. The situation with vertical resolution is simply this: Longer wavelengths, less resolution. The situation with horizontal resolution is similar, but the horizontal wavelength is directly measurable at the earth surface. Figure 3 shows this. Hyperboloids are shown from shallow and deep scatterers. Shallow hyperbolas have steep asymptotes. Deep scatterers have less steep asymptotes. So you see their horizontal wavelengths are longer. Thus you lose lateral spatial resolution with depth. Compounding the above reason for decreasing resolution is the loss of high frequency energy at late travel time.

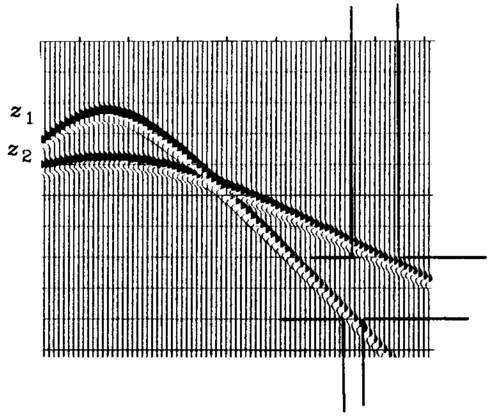


FIG. 3. Hyperboloids for an earth of velocity increasing with depth. Observable lateral wavelengths get longer with increasing depth. Thus lateral resolving power decreases with depth.

Two-Dimensional Fourier Transform

Before going any further, let us review some basic facts about two-dimensional Fourier transformation. A two-dimensional function is represented in a computer as numerical values in a matrix. A one-dimensional Fourier transform in a computer is an operation on a vector. A two-dimensional Fourier transform may be accomplished by a sequence of one-dimensional Fourier transforms. You may first transform each column vector of the matrix and then transform each row vector of the matrix. Alternately you may first do the rows and later do the columns. We can diagram the calculation as follows:

$$
\begin{array}{ccc}
p(t,x) & \longleftrightarrow & P(t,k_x) \\
\updownarrow & & \updownarrow \\
P(\omega,x) & \longleftrightarrow & P(\omega,k_x)
\end{array}
$$

An example of these transformations on some very typical deep ocean data is shown in figure 4.

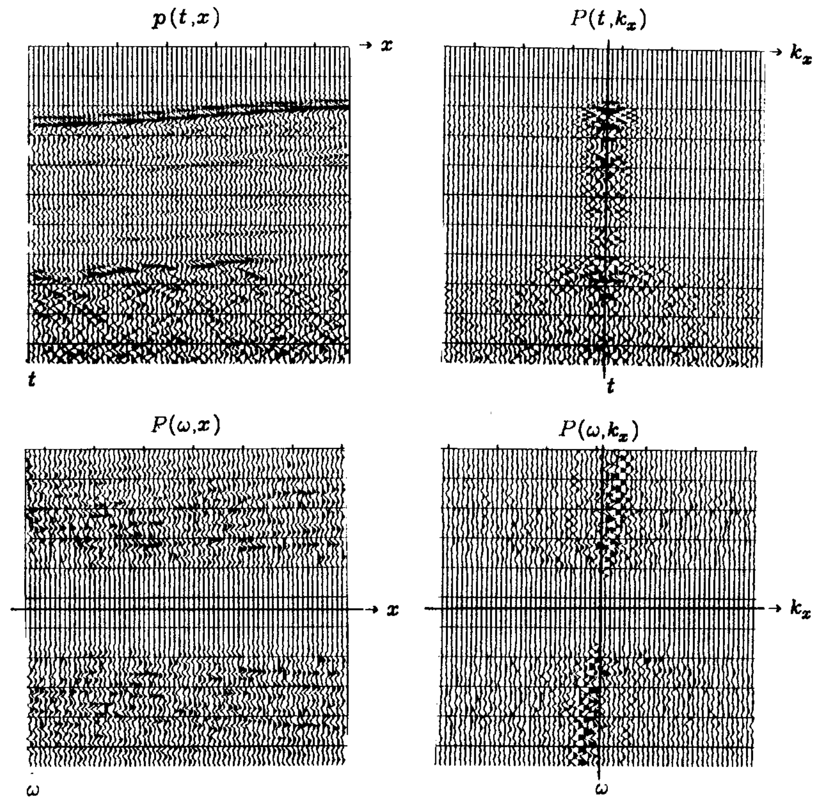


FIG. 4. A deep marine data set $p(t,x)$ and the real part of various Fourier transforms of it. Because of the long travel time through the water, the time axis does not begin at $t=0$.

In the deep ocean, sediments are fine grained and deposit slowly in flat, regular, horizontal beds. Lack of permeable rocks like sandstone severely reduces the potential for petroleum production from the deep ocean. The fine grained shales overlay irregular, igneous, basement rocks. The plot of $P(t,k_x)$ shows the lateral continuity of the sediments by the strong spectrum at low k_x. The igneous rocks show a k_x spectrum which drops off so slowly with k_x that the deep data are seen to be somewhat spatially aliased. The plot of $P(\omega, x)$ shows that the data contains no low frequency energy. At large ω the energy is not dropping off as fast as one might like which is indicative of temporal frequency aliasing. This aliasing is also apparent in the plot of $p(t, x)$ by the steplike appearance of the sea floor arrival. The dip of the seafloor shows up in (ω, k_x)-space by the energy crossing the origin at an angle.

A notational problem on the fore-mentioned diagram is that we cannot maintain the usual convention of using a lower case letter for the domain of physical space and an upper case letter for the Fourier domain, because the convention cannot include the mixed objects $P(t, k_x)$ and $P(\omega, x)$. Rather than invent some new notation it seems best to let the reader use the context to cope with this notational problem. The arguments of the function must help serve as the name of the function.

Altogether, the two-dimensional Fourier transform of a collection of seismograms involves only twice as much computation as the one-dimensional Fourier transform of each seismogram. This is lucky. Let us write a few equations to establish that the asserted procedure does indeed do a two-dimensional Fourier transform. First of all we express the idea that any function of x and t may be expressed as a superposition of sinusoidal functions

$$p(t, x) \quad = \quad \int\int e^{-i\omega t + i k_x x} \, P(\omega, k_x) \, d\omega \, dk_x \tag{9}$$

The kernel in this *inverse* Fourier transform has the form of a wave moving in the plus x-direction. Likewise, in the *forward* Fourier transform, the sign of both exponentials changes, preserving the fact that the kernel is a wave moving positively. The scale factor and the infinite limits are omitted as a matter of convenience. (The limits and scale both differ from the discrete computation, so why bother?) Now let us nest the double integration in a form which indicates that the temporal transforms are done first (inside):

$$p(t, x) \;=\; \int e^{i k_x x} \left[\int e^{-i\omega t} \, P(\omega, k_x) \, d\omega \right] dk_x \;=\; \int e^{i k_x x} \, P(t, k_x) \, dk_x$$

The quantity in brackets indicates temporal Fourier transforms being done for each and every k_x. Alternately, we could do the nesting with the k_x-integral on the inside. That would imply rows first instead of columns (or vice versa). It is the separability of $\exp(-i\omega t + i k_x x)$ into a product of exponentials which makes the computation this easy and cheap.

The Input-Output Relation

Let us return to the dispersion relation (8)

$$k_x^2 + k_z^2 \;=\; \frac{\omega^2}{v^2}$$

In applications where time evolves it is natural to solve (8) for $\omega(k_x, k_z)$. In extrapolation applications it is natural to solve for $k_z(\omega, k_x)$. Consider first an evolution situation. Any function which is a sinusoidal function of time may be evolved to future time t from initial conditions at t_0 by

$$p(x, z, t) \quad = \quad p(x, z, t_0) \; e^{-i\omega(t - t_0)} \tag{10a}$$

Setting $t_0 = 0$ and expressing the right hand side as a two dimensional inverse Fourier transform over space we get

$$p(x, z, t) \;=\; \int\int \left[P(k_x, k_z, t{=}0) \, e^{-i\omega(k_x,\, k_z) t} \right] e^{i k_x x + i k_z z} \, dk_x \, dk_z \tag{10b}$$

Setting $t{=}0$ in (10b) we see a double inverse Fourier transform which represents initial conditions in the (x, z)-plane. Taking $P(k_x, k_z, 0)$ to be constant would be a point source at $(x, z) = (0,0)$. The time-dependence in (10b) has been chosen [by selecting $\omega = \omega(k_x, k_z)$] to ensure that $p(x, z, t)$ is a wave field which fits the initial conditions at $t{=}0$.

Next consider the analogous wave-extrapolation situation.

$$p(z) \quad = \quad p(z_0) \; e^{i k_z (z - z_0)} \tag{11a}$$

Setting $z_0 = 0$ and incorporating a two dimensional Fourier transform over (x, t)-space we get

$$p(x, z, t) \;=\; \int\int \left[P(k_x, z{=}0, \omega) \; e^{i k_z(\omega, k_x) z} \right] e^{-i\omega t + i k_x x} \, d\omega \, dk_x \tag{11b}$$

At $z{=}0$ (11b) is just a double inverse Fourier transform which could represent geophysical observations in the (t, x)-plane at the earth's surface. The depth-dependence has been chosen [by selecting $k_z(\omega, k_x)$] to ensure that $p(x, z, t)$ is a wave field that matches the surface observations at $z{=}0$.

Equation (11b) is a prescription for finding waves inside the earth given waves observed at the earth's surface. We will return to it many times.

Notice that $\omega(k_x, k_z)$ and $k_z(\omega, k_x)$ are square-root functions and consequently there is a choice of signs. Initial conditions will determine what combination of the two solutions is desired. In the extrapolation case we have

$$k_z = \pm \left(\frac{\omega^2}{v^2} - k_x^2 \right)^{1/2} \tag{12a}$$

$$= \pm \frac{\omega}{v} \left(1 - \frac{v^2 k_x^2}{\omega^2} \right)^{1/2} \tag{12b}$$

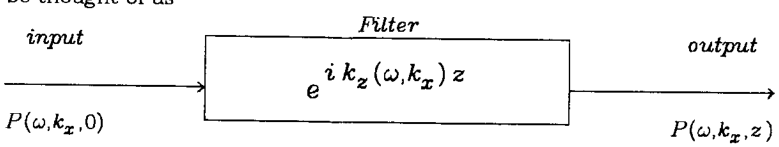
$$= \pm \frac{\omega}{v} \cos\vartheta \tag{12c}$$

Choice of the plus means that $\exp(-i\omega t + i k_z z)$ is a downgoing wave. The minus sign makes it upcoming (the usual case).

The quantity in brackets in (11) may be examined at any value of z. Given its value at one value of z, say $z=0$, we can determine its value at another. That is,

$$P(\omega, k_x, z) = P(\omega, k_x, 0)\, e^{i k_z(\omega, k_x) z} \tag{13}$$

This is a product relationship in both the ω-domain and the k_x-domain. Hence it can be regarded as a convolutional filter in t and x. In terms of engineering flow diagrams with inputs and outputs, equation (13) may be thought of as



What does this filter look like in the time and space domain? It turns out to be similar to delta function of $x^2 + z^2 - v^2 t^2$ which describes a cone. Physically, it is the Huygen's secondary wave source which was described in terms of ocean waves entering a gap through a storm barrier. Adding up the response of multiple gaps in the barrier would be convolution over x. Superposing many incident ocean waves would be convolution over t. (Mathematically, the exact inverse 2-D transform of the filter is a more tedious task, well beyond our present needs. As a practical matter the 2-D transforms are rather easy in a computer. Some slices of the cone are found in the section on programs).

The input-output filter, being of the form $e^{i\varphi}$, appears to be a phase shifting filter with ***no amplitude scaling***. This bodes well for our plans to deconvolve. It means that signal-to-noise power considerations will be much less relevant for migration than for ordinary filtration.

EXERCISES

1. Suppose that you are able to observe some shear waves at ordinary seismic frequencies. Is the spatial resolution better, equal, or worse than usual?
2. Explain the horizontal "layering" in figure 3 in the plot of $P(\omega, x)$. What determines the "layer" separation? What determines the "layer" slope?
3. Let $P(k_x, k_z)$ in (10b) be a constant signifying a point source at the origin in (x,z)-space. Let t be very large, meaning that the phase $= \varphi = [-\omega(k_x, k_z) + k_x(x/t) + k_z(z/t)]t$ in the integration is rapidly alternating with changes in k_x and k_z. Assume the only significant contribution to the integral comes when the phase is stationary. That is, where $\partial\varphi/\partial k_x$ and $\partial\varphi/\partial k_z$ both vanish. Where is the energy in (x,z,t)-space?