# Recursions for Migration in Slant Frames

*Bert Jacobs*

**Abstract**

Profiles require migration operators which handle large dips. One way of doing this is to use a high order migration operator. Another is to use a migration operator aimed to migrate dips centered at a predetermined angle, say in a slanted coordinate system. A simple recursion governs an infinite sequence of increasingly accurate slant migration operators. The first two terms of the sequence are the 15- and 45-degree equation operators.

## Introduction

The migration of common shot gathers might be preferred to stacking when moveout is strongly non-hyperbolic, when a point scatterer needs to be imaged. Migration might also be useful in the estimation of laterally varying velocity.

Common shot gathers have dips on them of anywhere from 0 to 90 degrees. Most of the dips present have the same sign. If this is in fact the case, it might be expedient to use a coordinate frame which reflects this fact. Migration operators in this frame will do the same job at less cost when compared to migration operators in the unslanted coordinate system.

We will restrict our attention to media which are laterally homogeneous. Phase shift methods are applicable in such media but have difficulty at boundaries. Finite difference techniques do not have these problems, so they will be used instead.

**Slant Coordinates**

The most fundamental parameter in a system of slant frames is the ray parameter $p$. We will consider media in which acoustic velocity $V$ is a function of depth, $z$, alone. In such media, $p$ is constant along raypaths. If the product $pv$ is identified with the sine of the propagation angle then we may define a new variable $s$ which is equal to the tan of the propagation angle.

$$s = \frac{pV}{\left(1-p^2 V^2\right)^{1/2}}$$

It is clear that $p$ must be restricted to values less than $1/V_{\max}$ in absolute value. With these definitions slant coordinates can be defined.

$$\begin{aligned} t' &= t - px \\ x' &= x - sz \\ z' &= z \end{aligned}$$

Applying these to the wave equation will require changes of variable in derivatives, too. Consider a differentiable function $Q$ and let $Q'(x',z',t') = Q(x,z,t)$ be the same field in the new coordinate system. Then

$$\begin{aligned} \frac{\partial Q}{\partial t} &= \frac{\partial Q'}{\partial t'} \\ \frac{\partial Q}{\partial x} &= \frac{\partial Q'}{\partial x'} - p\frac{\partial Q'}{\partial t'} \\ \frac{\partial Q}{\partial z} &= \frac{\partial Q'}{\partial z'} - s\frac{\partial Q'}{\partial x'} \end{aligned}$$

None of these derivatives have causal properties which will be required in the discretizations of a computer. Letting $D_x$ denote a causal $x$-derivative, it can be established that $-D_x^H$ is an anti-causal $x$-derivative. Similarly, $D_t$ is a causal $t$-derivative operator, $D_z$ is a causal $z$-derivative, $-D_t^H$ is an anti-causal $t$-derivative, and $-D_z^H$ is an anti-causal $z$-derivative.

The migration equation for a pressure wave field in a laterally homogeneous medium in an unslanted frame is given by

$$D_z^H Q = -\left[(-D_t^H)^2 + V D_x^H D_x V\right]^{1/2} \Lambda Q \qquad (1)$$

where $P$ is pressure, $\kappa$ is the bulk modulus of the medium, $Q = V^{1/2}P/\kappa^{1/2}$, and $\Lambda$ is the acoustic slowness of the medium. A migration equation in a slant coordinate system can be obtained by substituting according to a schedule derivable from the definition of the slant coordinates and the chain rule for partial differentiation. If we drop the primes, the schedule is as follows:

$$
\begin{aligned}
D_t &\leftarrow D_t \\
D_x &\leftarrow D_x - pD_t \\
D_z &\leftarrow D_z - sD_x
\end{aligned}
$$

The assignments indicated are not obvious and will be justified later. Substituting into equation (1) yields the slant migration equation

$$D_z^H Q = \left\{ sD_x^H - \left[ (-D_t^H)^2 + V(D_x^H - pD_t^H)(D_x - pD_t) V \right]^{1/2} \Lambda \right\} Q \tag{2}$$

This equation should be split in three parts to get an effective, accurate migrator. The three equations which must successively be solved at every $z$-step are

$$
\begin{aligned}
D_z^H Q &= -D_t^H \Lambda Q \\
D_z^H Q &= sD_x^H Q \\
D_z^H Q &= \left\{ D_t^H - \left[ (-D_t^H)^2 + V(D_x^H - pD_t^H)(D_x - pD_t) V \right]^{1/2} \right\} \Lambda Q
\end{aligned}
$$

Before proceeding further, it is expedient to work in the frequency domain. To do this, we replace $-D_t^H$ with $i\omega - \varepsilon_1$ or $i\omega - \varepsilon_2$, depending on location in the slant migration equation. The substitution is not arbitrary because we will still want the overall operator to be negative definite and will want $(D_x^H - pD_t^H)(D_x - pD_t)$ to be Hermitian.

$$
\begin{aligned}
-D_z^H Q &= (i\omega - \varepsilon_1) \Lambda Q \\
D_z^H Q &= sD_x^H Q \\
D_z^H Q &= \left\{ (i\omega - \varepsilon_1) - \left[ (i\omega - \varepsilon_1)^2 + V(D_x^H - p(-i\omega - \varepsilon_2))(D_x - p(i\omega - \varepsilon_2)) V \right]^{1/2} \right\} \Lambda Q
\end{aligned}
$$

Of these three equations, the first two can be solved analytically. Thus, the three way split is more or less equivalent to

$$
\begin{aligned}
Q(x, z+\Delta z, \omega) &= e^{(i\omega - \varepsilon_1)\Lambda \Delta z} \, Q(x, z, \omega) \\
Q(x, z+\Delta z, \omega) &= Q(x + s\Delta z, z, \omega) \\
D_z^H Q &= \left\{ (i\omega - \varepsilon_1) - \left[ (i\omega - \varepsilon_1)^2 + V(D_x^H - p(-i\omega - \varepsilon_2))(D_x - p(i\omega - \varepsilon_2)) V \right]^{1/2} \right\} \Lambda Q
\end{aligned}
$$

The last of these equations still needs to be discretized with respect to both $x$ and $z$ in a meaningful way. This will involve making approximations to the various derivatives in the differential equation.

**Derivative Approximation**

The simplest causal representation of the first $x$-derivative of a function $f$ is given by

$$D_x f \;=\; \frac{1}{\Delta x} B f$$

where $B$ is a causal operator. The matrix representation of $B$ has a diagonal of 1's and a subdiagonal of -1's and operates on a vector defined at discrete and regular intervals along the $x$-axis. The distinction between an operator and its representations is an important one because it allows us to use the same notation for continuous and discrete operations of similar nature.

If $B'$ has a triangular matrix representation with a diagonal and a subdiagonal of 1's then another causal approximation to the $x$-derivative can be written as

$$D_x f \;=\; \frac{2}{\Delta x} (B')^{-1} B f$$

This $x$-derivative of $f$ is equivalent to the the Crank-Nicolson approximation and is the one used for the depth derivative in migration problems.

A rational, causal $x$-derivative for $f$ can be formulated by setting

$$D_x f \;=\; \frac{1}{\Delta x} B \left[ I - \frac{\alpha}{1+\alpha} B \right]^{-1} f$$

where $\alpha$ is a real and positive number. The advantage in using this derivative is that it leads to an unusually good implementation for second $x$-derivatives. With a little algebra it can be proved that

$$-D_x^H D_x f \;=\; \frac{1}{(\Delta x)^2} B B^H \left[ I - \frac{\alpha}{(1+\alpha)^2} B B^H \right]^{-1} f$$

The operators $B$ and $B^H$ have some useful algebraic properties that will be used later. The most important of these is that $BB^H = B^H B = B^H + B$. Since $B$ and $B^H$ commute

$$D_x \pm D_x^H \;=\; (B \pm B^H) \left[ \frac{\alpha}{(1+\alpha)^2} B B^H \right]^{-1}$$

This result will be used in the slant migration equation when the dissipation parameters are independent of $x$. Since it has already occurred several times, $BB^H$ is important enough to merit its own symbol. Thus, we set $BB^H = T$, where $T$ is an operator whose matrix representation has 2's on its diagonal and -1's on both its superdiagonal and subdiagonal. Note that $T$ is real, symmetric, and non-negative.

**The Slant Focusing Equation**

The slant focusing equation is the third equation in our three-way split of the slant migration equation. Copying this equation for convenience,

$$D_z^H Q = \left\{(i\omega-\varepsilon_1)-\left[(i\omega-\varepsilon_1)^2+V(D_x^H-p(-i\omega-\varepsilon_2))(D_x-p(i\omega-\varepsilon_2))V\right]^{1/2}\right\}\Lambda Q$$

we substitute approximations for $D_x^H D_x$, $D_x - D_x^H$, and $D_x + D_x^H$. The result of this substitution and the continued fractions which follow can be simplified by introducing a constant $\beta = \alpha/(1+\alpha)$ and two more operators $n$ and $d$.

$$d = I-\beta T \tag{3}$$

$$n = \frac{(\Delta z)^2}{4}\left[(\Delta x)^{-2}T+p^2(\omega^2-\varepsilon_2^2)(I-\beta T)+p(\Delta x)^{-1}\varepsilon_2(B+B^H)+ip(\Delta x)^{-1}\omega(B-B^H)\right] \tag{4}$$

If $\varepsilon_2$ is independent of $x$ the operators $n$ and $d$ commute with one another. The dissipation parameter $\varepsilon_1$ will be required to be $x$-independent as well. With all these conventions and definitions, the resulting differential equation is

$$-D_z^H Q = -\left\{(i\omega-\varepsilon_1)-\left[(i\omega-\varepsilon_1)^2+Vnd^{-1}V\right]^{1/2}\right\}\Lambda Q \tag{5}$$

which we choose to call the slant focusing equation. Applying a little algebra and using the Crank-Nicolson derivative in equation (5) we get

$$(I-Op)Q(x,z+\Delta z,\omega) = (I+Op)Q(x,z,\omega) \tag{6}$$

$$Op = -\frac{\Delta z}{2}\Lambda(i\omega-\varepsilon_1)+\left[\left(\frac{\Delta z}{2}\right)^2\Lambda^2(i\omega-\varepsilon_1)^2+\left(\frac{\Delta z}{2}\right)^2 nd^{-1}\right]^{1/2}$$

which has a vector space representation which is a linear system of equations.

The right hand side of equation (6) has an operator with a square root in it. The square root should be expanded in a continued fraction and the continued fraction truncated to get a useful migration scheme. Thus $Op$ is equal to

$$\cfrac{I}{\frac{1}{2}(\Delta z\Lambda(i\omega-\varepsilon_1))^2+\cfrac{I}{\frac{1}{2}(\Delta z\Lambda(i\omega-\varepsilon_1))^2+\cfrac{I}{\frac{1}{2}(\Delta z\Lambda(i\omega-\varepsilon_1))^2+\cdots}nd^{-1}}nd^{-1}}nd^{-1}$$

The $d^{-1}$'s are an annoyance that can be done without. Employing a similarity transformation to the continued fraction, the right partial numerator can be cleared of them. The result is a fraction in which every partial numerator as well as every other partial denominator is equal. Such a continued fraction is said to be periodic with a period of two.

$$\cfrac{I}{\frac{1}{2}(\Delta z\,\Lambda(i\omega-\varepsilon_1))^2 d+\cfrac{I}{\frac{1}{2}(\Delta z\,\Lambda(i\omega-\varepsilon_1))^2+\cfrac{I}{\frac{1}{2}(\Delta z\,\Lambda(i\omega-\varepsilon_1))^2 d+\cdots}-n}-n}-n$$

This fraction has a sequence of approximants and each approximant has a numerator. Thus, there exist sequences of numerators for the approximants of the two operators on either side of equation (6). The numerator of the $k$th approximant of the operator on the left and right sides of equation (6) will be denoted by $A_k^D$ and $A_k^N$, respectively. The denominators of the approximants of the operators on the two sides of equation (6) cancel one another, so they will not be discussed in this paper. It follows that the linear equation that we want to solve is

$$A_k^D Q(x,z+\Delta z,\omega) \;=\; A_k^N Q(x,z+\Delta z,\omega)$$

for $k$ large enough to be accurate for all dips present. Setting $k = 1$ will yield the 15-degree equation while the 45-degree equation can be obtained when $k = 2$. The operators $A_k^N$ and $A_k^D$ obey similar recurrence relations. Dropping the superscript for compactness we get a dimensionless recursion for the two operators

$$\begin{aligned}
A_{-1} &= I \\
A_0 &= I \\
A_1 &= \pm nA_{-1} + \frac{1}{2}(\Delta z\,\Lambda(i\omega-\varepsilon_1))^2 dA_0 && \text{- for D, + for N} \\
A_{k+1} &= nA_{k-1} + \frac{1}{2}(\Delta z\,\Lambda(i\omega-\varepsilon_1))^2 A_k && \text{k odd} \\
A_{k+1} &= nA_{k-1} + \frac{1}{2}(\Delta z\,\Lambda(i\omega-\varepsilon_1))^2 dA_k && \text{k even}
\end{aligned}$$

in terms of the simple operators $n$ and $d$, defined in equations (3) and (4).

### Appendix A - Continued Fractions with Matrix Coefficients

The migration problem has generated a continued fraction with matrix operators for coefficients. The algebra of these fractions needs to be developed a bit before we proceed much further. A continued fraction generates a sequence of rational forms called approximants. For any continued fraction, we will need a quick algorithm for generating its sequence of approximants, rules for changing the coefficients of the continued fraction in such a way as to leave the sequence of approximants untouched, and an understanding of how the properties of matrices and continued fractions interact.

Given coefficients, each an $N$ by $N$ matrix,

$$\left\{ a_j \right\}_{j=1}^{\infty}, \quad \left\{ b_j \right\}_{j=0}^{\infty}, \quad \left\{ c_j \right\}_{j=1}^{\infty}$$

a continued fraction can be generated. The following argument will be formal, in that the existence of the necessary inverses and the convergence of the continued fraction will be assumed rather than proved. With this understood, a continued fraction which we denote $F$ will be considered, where equal to

$$F = b_0 + a_1 \frac{I}{b_1 + a_2 \dfrac{I}{b_2 + a_3 \dfrac{I}{b_3 + \cdots} c_3} c_2} c_1$$

This continued fraction can be considered to be the resultant of a series of non-linear transformations $t_p$, where $p$ varies from 0 to $\infty$. This sequence of transformations is defined on an $N$ by $N$ matrix argument $w$ and takes the form

$$t_0(w) = b_0 + w$$

$$t_p(w) = a_p \frac{I}{b_p + w} c_p \qquad p = 1,2,3,\cdots$$

Transformations of this type can be combined via the operation of functional composition. For example, the resultant of operating on $t_1$ with $t_0$ is a transformation

$$t_0 t_1(w) = b_0 + a_1 \frac{I}{b_1 + w} c_1$$

Similarly, our continued fraction $F$ can be considered to be the limit of a sequence of transformational compositions evaluated with some particular value of $w$. If $F$ is well-defined then the choice of $w$ will not matter.

$$F = F(w) = \lim_{p \to \infty} t_0 t_1 t_2 \cdots t_p(w)$$

A limit is not useful for computations because limits usually are infinitely expensive to compute. In migration applications it will turn out to be useful to consider intermediate terms of the limiting sequence. The most important result in this section, to be proved by induction in a separate section at the end, is that

$$F_k(w) = t_0 t_1 t_2 \cdots t_k(w) = \left[B_k + w a_k^{-1} B_{k-1}\right]^{-1} \left[A_k + w a_k^{-1} B_{k-1}\right]$$

$$A_{-1} = I \qquad A_0 = b_0 \qquad a_0 = I$$

$$B_{-1} = 0 \qquad B_0 = I$$

$$A_{k+1} = c_{k+1} a_k^{-1} A_{k-1} + b_{k+1} a_{k+1}^{-1} A_k \qquad k = 1,2,3,...$$

$$B_{k+1} = c_{k+1} a_k^{-1} B_{k-1} + b_{k+1} a_{k+1}^{-1} B_k \qquad k = 1,2,3,...$$

Now for some nomenclature: $A_n$ is the $n$th numerator, $B_n$ is the $n$th denominator, the ratio $B_n^{-1} A_n$ is the $n$th approximant, $a_n$ is the $n$th left partial numerator, $c_n$ is the $n$th right partial numerator, and $b_n$ is the $n$th partial denominator. The difference from the usual continued fraction theory lies in the distinction between left and right partial numerators.

There exist an infinite number of continued fractions with the same value and the same series of approximants as $F$. Consider two sets of $N$ by $N$ matrices

$$\left\{ d_j \right\}_{j=1}^{\infty}, \qquad \left\{ e_j \right\}_{j=1}^{\infty}$$

where the $d_j$ are all invertible. One continued fraction that has the same approximants as $F$ can be obtained by simultaneously pre-multiplying $c_1$, $b_1$, and $a_2$ by an $N$ by $N$ matrix $e_1$. This pattern of matrix multiplication is used because the pattern of matrices, ignoring $b_0$ for the moment, is basically $a_1 M^{-1} c_1$ where $M$ contains all of the rest of the continued fraction. Premultiplying $c_1$ by $e_1^{-1} e_1$ yields $a_1 M_{-1} e_1^{-1} e_1 c_1$ which is equivalent to $a_1 (e_1 M)^{-1} e_1 c_1$. If we look back at the fundamental recurrence formulae for $A_k$ and $B_k$ then it can be seen that this transformation leaves all approximants of the continued fraction unchanged. The result of this transformation is the continued fraction

$$b_0 + a_1 \cfrac{I}{e_1 b_1 + e_1 a_2 \cfrac{I}{b_2 + a_3 \cfrac{I}{b_3 + \cdots} c_3} c_2} e_1 c_1$$

A transformation on the coefficients of a continued fraction that preserves the sequence of approximants will be called an equivalence transformation. A much more general equivalence transformation of $F$ yields the following continued fraction:

$$b_0 + a_1 d_1 \cfrac{I}{e_1 b_1 d_1 + e_1 a_2 d_2 \cfrac{I}{e_2 b_2 d_2 + e_2 a_3 d_3 \cfrac{I}{e_3 b_3 d_3 + \cdots} e_3 c_3 d_2} e_2 c_2 d_1} e_1 c_1$$

**Appendix B - The Fundamental Recurrence for Continued Fractions**

We have defined a sequence of rational transformations $t_p$, where $p$ varies between 0 and $\infty$ and

$$t_0(w) = b_0 + w$$

$$t_p(w) = a_p \frac{I}{b_p + w} c_p$$

It would be desirable to find a recurrence formula for the functional compositions $t_0 t_1 t_2 \cdots t_k(w)$. Suppose the $k$th composition is of the form

$$F_k(w) = \left[B_k + w a_k^{-1} B_{k-1}\right]^{-1} \left[A_k + w a_k^{-1} A_{k-1}\right]$$

Then

$$F_{k+1}(w) = F_k t_{k+1}(w) = F_k(a_{k+1} \frac{I}{b_{k+1} + w} c_{k+1})$$

With a little algebra, paying close attention to the lack of commutativity among the various matrices, this expression can be simplified to look like

$$F_{k+1}(w) = \left[(c_{k+1} a_k^{-1} B_{k-1} + b_{k+1} a_{k+1}^{-1} B_k) + w a_{k+1}^{-1} B_k\right]^{-1} \left[(c_{k+1} a_k^{-1} A_{k-1} + b_{k+1} a_{k+1}^{-1} A_k) + w a_{k+1}^{-1} A_k\right]$$

$$F_{k+1}(w) = \left[B_{k+1} + w a_{k+1}^{-1} B_k\right]^{-1} \left[A_{k+1} + w a_{k+1}^{-1} A_k\right]$$

Equating coefficients yields a recurrence for both the $A_k$'s and $B_k$'s in terms of the partial numerators and denominators:

$$\begin{aligned} A_{k+1} &= c_{k+1} a_k^{-1} A_{k-1} + b_{k+1} a_{k+1}^{-1} A_k \\ B_{k+1} &= c_{k+1} a_k^{-1} A_{k-1} + b_{k+1} a_{k+1}^{-1} B_k \end{aligned}$$

The necessary initializations for this recurrence need to be found. To get the starting points, consider the cases in which $k = 0$ and $k = 1$.

$$t_0(w) = b_0 + w = \left[I + 0w\right]^{-1} \left[b_0 + Iw\right]$$

$$t_1(w) = (b_1 a_1^{-1} + w a_1^{-1})^{-1} (b_1 a_1^{-1} b_0 + c_1 + w a_1^{-1} b_0)$$

Equating coefficients again, we find that for non-zero $a_0$

$$\begin{aligned} A_{-1} &= a_0 \quad & A_0 &= b_0 \\ B_0 &= I \quad & B_0 &= I \end{aligned}$$

will provide a suitable recurrence initialization. For convenience, we set $a_0$ and therefore $A_{-1}$ equal to identity operators.

From the fundamental recurrence it can be seen that pre-multiplying $b_{k+1}$ and $c_{k+1}$ by the same non-singular matrix will not change the approximants of the continued fraction. The same can be said for post-multiplication of $c_{k+1}$ and $a_k$ and for post-multiplication of $b_{k+1}$ and $a_{k+1}$. These facts were used implicitly in Appendix A in the discussion of equivalence transformations.

**Bibliography**

**Engquist, Bjorn, Well-Posedness of One Way Wave Equations, SEP-8, pp. 48-69.**

**Brown, D., One-Way Wave Equations by Factorization into Pseudo-Differential Operators for the Variable Coefficient Case, SEP-13, pp. 41-6.**

**Godfrey, B., Muir, F., and Claerbout, J.F., Stable Extrapolation, SEP-16, pp. 83-8.**

**Jacobs, A., and Muir, F., High Order Migration Operators for Laterally Homogeneous Media, SEP-26, pp. 163-82.**

**Clayton, R., Stable and Accurate Extrapolation Operators in an Acoustic Medium, SEP-27, pp.67-79.**

**Wall, H.S., (1973), Analytic Theory of Continued Fractions: Bronx, N.Y., Chelsea Publishing Co.**