# Differential Geometry and Ray-Centered Coordinates

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### Frénet's Formulae

In the study of ray tracing it is useful to attach particular coordinate systems to a specific ray. This entails a rather intimate knowledge of the differential geometry of space curves (rays).

Let us begin by parameterizing our ray by the arc length $S$ . Then $\mathbf{X}(S)$ is a vector from the origin to that point on the ray located by any value of $S$ . Consider now the tangent vector,

$$\mathbf{l} = \frac{d\mathbf{X}(S)}{dS} \tag{3-1}$$

It is clear that $\mathbf{l}$ is a unit vector, since

$$\mathbf{l}\cdot\mathbf{l} = \frac{d\mathbf{X}(S)}{dS}\cdot\frac{d\mathbf{X}(S)}{dS} = 1$$

Immediately it follows that

$$\mathbf{l}\cdot\frac{d\mathbf{l}}{dS} = 0 \quad ,$$

or $\mathbf{l}$ is perpendicular to $d\mathbf{l}/dS$ . The vector $d\mathbf{l}/dS$ is not a unit vector. Its magnitude is the curvature $\kappa(S)$ , and its direction will be denoted by the unit normal vector $\mathbf{n}$ , i.e.

$$\frac{d\mathbf{l}}{dS} = \kappa(S)\,\mathbf{n} \tag{3-2}$$

The unit vectors $\mathbf{l}$ and $\mathbf{n}$ determine a plane known as the osculating plane which, in simple ray optics, coincides with the plane of the ray. Otherwise, the osculating plane changes direction and tends to twist as we move along the ray. This necessitates the introduction of a third unit vector, the binormal unit vector, denoted by $\mathbf{b}$ where $\mathbf{b} = \mathbf{l}\times\mathbf{n}$ . Our differential geometry picture is almost complete now. We have defined an orthogonal triad of unit vectors **l**, **n** and **b**. What is missing is $d\mathbf{b}/dS$ and $d\mathbf{n}/dS$.

To calculate $d\mathbf{b}/dS$ first, we note that

$$\frac{d\mathbf{b}}{dS} = \frac{d\mathbf{l}}{dS}\times\mathbf{n} + \mathbf{l}\times\frac{d\mathbf{n}}{dS} \tag{3-3}$$

From (3-2), the first term in (3-3) vanishes. Since **n** is a unit vector, $d\mathbf{n}/dS$ must be a linear combination of **l** and **b**. Thus $\mathbf{l}\times d\mathbf{n}/dS$ is proportional to **n**. This constant of proportionality is called the torsion $T$. Therefore,

$$\frac{d\mathbf{b}}{dS} = -\,T(S)\mathbf{n} \tag{3-4}$$

where a negative sign has been introduced so that a positive twist will correspond to a counterclockwise rotation. From the similarity between (3-2) and (3-4), $T(S)$ is also known as the second curvature.

From the previous definitions, it is easy to calculate $d\mathbf{n}/dS$:

$$\frac{d\mathbf{n}}{dS} = \frac{d\mathbf{b}}{dS}\times\mathbf{l} + \mathbf{b}\times\frac{d\mathbf{l}}{dS}$$

$$= -\,T(S)\,(\mathbf{n}\times\mathbf{l}) + \kappa(S)\,(\mathbf{b}\times\mathbf{n})$$

$$= T(S)\mathbf{b} - \kappa(S)\mathbf{l} \tag{3-5}$$

Equations (3-2), (3-4) and (3-5) are the Frénet formulae, and will be useful in all future discussions concerning ray centered coordinate systems.

## Ray-Centered Coordinates

Suppose we have traced a particular ray. It is parameterized by its arc length $S$, and we shall define its location with respect to a fixed origin as the vector $\mathbf{r}(S)$, or simply $\mathbf{r}_o$. To consider neighboring rays, we shall introduce two new coordinates $q_1$ and $q_2$. These coordinates will define our location on a plane perpendicular to the tangent vector **l**. Thus, a vector **r** to a point near the ray we have traced (the central ray) can be defined as

$$\mathbf{r}(S,q_1,q_2) = \mathbf{r}_o + q_1\,\mathbf{e}_1 + q_2\,\mathbf{e}_2 \tag{3-6}$$

with $\mathbf{e}_1$ and $\mathbf{e}_2$ as special unit vectors defined in the plane perpendicular to the ray unit tangent vector **l**. How can we define $\mathbf{e}_1$ and $\mathbf{e}_2$?

The unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ will be a linear combination of $\mathbf{n}$ and $\mathbf{b}$ defined as follows:

$$\mathbf{e}_1 = \mathbf{n}\cos\vartheta - \mathbf{b}\sin\vartheta \quad \text{and} \quad \mathbf{e}_2 = \mathbf{n}\sin\vartheta + \mathbf{b}\cos\vartheta \tag{3-7}$$

The quantity $\vartheta$ will be chosen in a special way. It will be fixed so that both $d\mathbf{e}_1/dS$ and $d\mathbf{e}_2/dS$ will not depend on $\mathbf{e}_1$ and $\mathbf{e}_2$, but only on $\mathbf{l}$, the unit tangent vector. This means that the local triad $\mathbf{l}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ will not twist as we move along the ray. We are, in essence, "freezing" the triad if $\vartheta$ is picked properly.

First, we calculate $d\mathbf{e}_1/dS$ and use the Frénet formulae:

$$\frac{d\mathbf{e}_1}{dS} = \frac{d\mathbf{n}}{dS}\cos\vartheta - \frac{d\mathbf{b}}{dS}\sin\vartheta - \mathbf{n}\sin\vartheta\,\frac{d\vartheta}{dS} - \mathbf{b}\,\cos\vartheta\,\frac{d\vartheta}{dS}$$

$$= \cos\vartheta\Big[T(S)\mathbf{b} - \kappa(S)\mathbf{l}\Big] + T(S)\,\sin\vartheta\mathbf{n} - \mathbf{n}\sin\vartheta\,\frac{d\vartheta}{dS} - \mathbf{b}\cos\vartheta\,\frac{d\vartheta}{dS} \tag{3-8}$$

It is clear that if $d\mathbf{e}_1/dS$ is to depend only on the direction of $\mathbf{l}$ then

$$\frac{d\vartheta}{dS} = T(S) \qquad or \qquad \vartheta(S) = \int_{S_1}^{S} T(S)dS + \vartheta(S_1) \tag{3-9}$$

where $S_1$ denotes the beginning of the ray. Thus, for $d\mathbf{e}_1/dS$ and $d\mathbf{e}_2/dS$ we have

$$\frac{d\mathbf{e}_1}{dS} = -\kappa(S)\,\cos\vartheta\,\mathbf{l} \quad \text{and} \quad \frac{d\mathbf{e}_2}{dS} = -\kappa(S)\,\sin\vartheta\,\mathbf{l} \tag{3-10}$$

where $\vartheta(S) = \int_{S_1}^{S} T(S)dS + \vartheta(S_1)$ and $\kappa(S)$ is the ray curvature. The important point to note about (3-10) is that $d\mathbf{e}_1/dS$ or $d\mathbf{e}_2/dS$ depend on $S$ only.

The above derivation is cardinal to the development of certain ray approximations. With the above definitions, for example, we can calculate the local arc length as follows. From (3-6)

$$d\mathbf{r} = \left(\frac{d\mathbf{r}_o}{dS} + q_1\,\frac{d\mathbf{e}_1}{dS} + q_2\,\frac{d\mathbf{e}_2}{dS}\right)dS + \mathbf{e}_1\,dq_1 + \mathbf{e}_2\,dq_2 \tag{3-11}$$

Use of the facts that $d\mathbf{r}_o/dS = \mathbf{l}$, (3-10), and that $\mathbf{l}$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are an orthogonal triad gives the result that

$$d\mathbf{r}\cdot d\mathbf{r} = h^2 dS^2 + dq_1^2 + dq_2^2 \tag{3-12}$$

where $h = 1 - \kappa(q_1\,\cos\vartheta + q_2\,\sin\vartheta)$, with $\vartheta$ defined as before. From (3-12), we immediately know that the scale factors for our ray-centered coordinate system are

$h$, 1, 1 . Now we can define local gradient operators, based on these factors. This allows us to obtain directly the wave equation in the ray-centered coordinate system.