Chapter VI: The Choice of Variables For Elastic Wave Extrapolation

Abstract

The choice of variables for elastic problems determines the form of the one-way extrapolation equations. Three sets of variables are considered: displacements, potentials, and a mixed set of variables which eliminate troublesome ∂_{zz} terms from the full wave equation. The displacements satisfy boundary conditions at internal interfaces but presently lack a recurrence relation to generate higher order approximation. The potential variables (P and S waves) appear to have problems similar to those of the displacements for variable velocity media, because the equation contains a complicated term which is difficult to factor into up- and downgoing waves. The mixed set of variables have the desired recurrence relation but fail to satisfy the boundary conditions. They also appear to be unstable for variable medium parameters.

6.1 Introduction

In this chapter the choice of state variables for the modeling of elastic waves by extrapolation with one-way wave equations is discussed. One use of such extrapolation operators is in the migration and inversion of elastic wavefields. The main concern in migration is that the reflectors be placed in their correct positions, and usually the accuracy of the amplitudes is not of primary importance. However, with a little more care in the way medium variations are incorporated into the extrapolation operators, one should be able to achieve accurate amplitudes as well as accurate traveltimes. This means that one may consider using extrapolation methods to provide solutions to forward modeling problems, such as refracted body waves and surface waves. In Chapter V, the extrapolation of scalar wavefields was presented. Our goal is to extend the scalar methods to elastic wave problems. The first step is to settle on a set of state variables for the problem. With the scalar wave equation, the question of the choice of state variables does not arise because the wave equation is already in its simplest form. For acoustic waves a pressure (or potential) variable is used, and for SH-waves a displacement variable is used.

For elastic extrapolation there are several choices of variables. The first is the displacements themselves, which are governed by a coupled vector wave equation. For the constant velocity elastic wave equation, the most obvious choice of variables are the potential variables (P and S waves), which convert the coupled vector equations into a pair of uncoupled scalar problems. Another choice is a set of mixed variables ("mixed" for lack of a better name) that transform the equation into one that looks like a scalar wave equation with matrix coefficients. In the sections that follow, we will discuss the relative merits of each type of state variable.

6.2 Displacement Variables

In elastic theory three types of variables are usually discussed: displacements, stresses, and potentials. If one also happens to be interested in differential operators, then displacements will be the fundamental set because both stresses and potentials are expressible as first order differentials of displacements. Thus, it is particularly easy to transform the displacement solution into stresses to apply boundary conditions, or into potentials (P and S waves) for interpretation.

The problem with displacements is that the full operator (for constant density)[1]

$$(\partial_z A\, \partial_z + \partial_z B\, \partial_x + \partial_x B^T \partial_z + \partial_x C\, \partial_x + \omega^2 I)u = 0 \tag{6.1}$$

where

$$u = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} \text{horzontal displacement} \\ \text{vertical displacement} \end{bmatrix}$$

and

$$A = \begin{bmatrix} \beta^2 & 0 \\ 0 & \alpha^2 \end{bmatrix},\ B = \begin{bmatrix} 0 & \beta^2 \\ \alpha^2 - 2\beta^2 & 0 \end{bmatrix},\ \text{and } C = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}$$

contains both first and second order differentials in z. Consequently, the formal factoring of the operator into one-way equations is

$$(\partial_z + D_1)(\partial_z - D_2)u = 0. \tag{6.2}$$

If the first order differentials were absent in equation (6.1), then D_1 would equal D_2, and the situation would be a lot simpler. Expanding equation (6.2) and matching it to a constant velocity form of equation (6.1) leads to the constraints

$$D_1 - D_2 = iA^{-1}(B + B^T)k_x$$

and

[1] In this chapter ∂_x and ∂_z denote partial derivatives with respect to x and z.

$$D_1 D_2 = A^{-1}(Ck_x^2 - I\omega^2)$$

Solving for D_1 and D_2 we have the quadratic equations

$$D_1^2 - ik_x D_1 A^{-1}(B + B^T) + A^{-1}(I\omega^2 - Ck_x^2) = 0 \tag{6.3}$$

and

$$D_2^2 + ik_x A^{-1}(B + B^T) D_2 + A^{-1}(I\omega^2 - Ck_x^2) = 0. \tag{6.4}$$

Thus, in order to build up a sequence of approximations for one-way displacement operators, the recurrence relation has to solve a quadratic rather than the simpler square root of scalar theory [c.f. equation (4.9)]. We have not been successful in finding such a recurrence relation. The chief difficulty seems to be that the D matrices in equations (6.3) and (6.4) do not commute with either A, B, or C.

At present there exists exact expressions in the (z, k_x, w)–domain for the operators D_1 and D_2

$$D_1 = \frac{i}{k_x^2 + \nu_\alpha \nu_\beta} \begin{bmatrix} \dfrac{\omega^2 \nu_\alpha}{\beta^2} & k_x \nu_\beta(\nu_\alpha - \nu_\beta) \\ k_x \nu_\alpha(\nu_\alpha - \nu_\beta) & \dfrac{\omega^2 \nu_\beta}{\alpha^2} \end{bmatrix}$$

and

$$D_2 = \frac{i}{k_x^2 + \nu_\alpha \nu_\beta} \begin{bmatrix} -\dfrac{\omega^2 \nu_\alpha}{\beta^2} & k_x \nu_\beta(\nu_\alpha - \nu_\beta) \\ k_x \nu_\alpha(\nu_\alpha - \nu_\beta) & -\dfrac{\omega^2 \nu_\beta}{\alpha^2} \end{bmatrix} \tag{6.5}$$

where $\nu_\alpha = \sqrt{\omega^2/\alpha^2 - k_x^2}$ and $\nu_\beta = \sqrt{\omega^2/\beta^2 - k_x^2}$. A second order approximation to the extrapolation operators can be obtained by expanding the square roots in equation (6.5) in a 2^{nd}-order Taylor series (Clayton and Engquist, 1977). However, from scalar theory we know that higher order approximations obtained by further Taylor series expansion will lead to unstable operators.

If a recurrence relation for displacement variables seems so difficult to obtain, the obvious question is why bother with them? The answer is that of all the variables considered in this chapter, they come the closest to satisfying the internal boundary conditions. The boundary conditions for an elastic medium are continuity of stresses and displacements. For an interface parallel to the z–axis, these may be written as (for constant density)

$$\left[\begin{pmatrix} \alpha^2\partial_z & (\alpha^2-2\beta^2)\partial_x \\ \beta^2\partial_x & \beta^2\partial_z \end{pmatrix} u \right] = 0 \text{ and } \left[u \right] = 0. \tag{6.6}$$

where the square brackets denote differences across the interface.

For a plane wave traveling in z-direction, the dominant term in the one-way approximation, as far as the boundary conditions are concerned, is $E\partial_{zz}$, where E is determined by matching the dispersion relation. If we write this as $E_1\partial_z E_2\partial_z$ the implicit boundary conditions are

$$\left[u\right] = 0 \text{ and } \left[E_2\partial_z u\right] = 0.$$

Thus if we choose $E_1 = E_2^{-1}E$ and

$$E_2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix}$$

we will satisfy the correct boundary conditions [equation (6.6) with $\partial_x = 0$]. If the wave impinges at some other angle of incidence, then other terms in the one-way operator become significant. This case has not yet been analyzed.

In summary, displacement variables lack a recurrence relation to generate higher order approximations. However, since the stresses can be obtained from first order differentials of the displacements, they have the potential for matching the implicit boundary conditions.

6.3 Potential Variables

The displacement wave equation in a constant velocity medium can be diagonalized (decoupled) by introducing potentials of either the form

$$u = \nabla P + \nabla \times S,$$

or the form

$$P' = \nabla \cdot u \text{ and } S' = \nabla \times u .$$

The two are equivalent in the sense that

$$P' = -\frac{\omega^2}{\alpha^2}P \text{ and } S' = -\frac{\omega^2}{\beta^2}S.$$

Consider writing equation (6.1) in the form

$$(H - J)u = 0$$

where H is the homogeneous part (or constant velocity part) and J contains all the velocity variations. Conversion of this equation to P and S potentials amounts to diagonalizing the operator H. Defining the diagonalization as

$$Q^{-1}H\,Q = (\partial_{zz} - \Lambda^2) \text{ and } p = \begin{bmatrix} P \\ S \end{bmatrix} = Q^{-1}u,$$

the equation becomes

$$(\partial_{zz} - \Lambda^2 - Q^{-1}J\,Q)p = 0.$$

We may formally factor the problem as

$$(\partial_z + \Lambda)(\partial_z - \Lambda)\,p = Q^{-1}J\,Q\,p. \tag{6.7}$$

The right-hand side can be considered as a source term for the homogeneous operator on the left. The problem is sorting out what parts of the source term to retain. It is obvious, by the way the operator is factored, that the left-hand side contains waves moving in both directions in z. What is not so obvious is that the right hand side also does. It contains reflection and transmission coefficients for both types of waves. It is important to eliminate the backscattering components from the source term. For example, if we inadvertently retain a reflection term for a backscattered wave, the extrapolation operator will start a new wave moving in the extrapolation direction.

The operator $Q^{-1}J\,Q$ is more complicated than the displacement equation itself. Thus, by transforming to P and S waves, it would appear that the problem has become more complicated.

6.4 Mixed Variables

The equations of motion for a two-dimensional elastic medium in Cartesian coordinates can be written as five first-order partial differential equations in the variables $(1/\rho)\tau_{xx}$, $(1/\rho)\tau_{zz}$, $(1/\rho)\tau_{xz}$ (normalized stresses) and u, w (horizontal and vertical displacements) (Claerbout, 1976, p.182). By eliminating the variable $(1/\rho)\tau_{xx}$ and differentiating two of the resulting equations by x, these equations can be written as a 4 by 4 system of equations in the following form:

$$\frac{\partial}{\partial z}\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} \tag{6.8}$$

where $r = [\,\partial_x u\,,\,(1/\rho)\tau_{zz}\,]^T$, $s = [\,w\,,\,(1/\rho)\partial_x\tau_{xz}\,]^T$, and the matrix operators A and B are given by

$$A = \begin{bmatrix} -\partial_{xx} & \frac{1}{\beta^2} \\ \partial_{tt} & -1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{2\beta^2-\alpha^2}{\alpha^2} & \frac{1}{\alpha^2} \\ \partial_{tt} - \frac{4\beta^2}{\alpha^2}(\alpha^2-\beta^2)\partial_{xx} & \frac{2\beta^2-\alpha^2}{\alpha^2}\partial_{xx} \end{bmatrix} \qquad (6.9)$$

$\partial_{zz} q = (M_1 \partial_{tt} + M_2 \partial_{xx}) q$			
q	M_1	M_2	Q
$\begin{bmatrix} \partial_z w \\ \frac{1}{\rho}\tau_{zz} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\alpha^2} & 0 \\ -2\frac{\alpha^2-\beta^2}{\alpha^2} & \frac{1}{\beta^2} \end{bmatrix}$	$\begin{bmatrix} \frac{\alpha^2-2\beta^2}{\alpha^2} & -\frac{\alpha^2-\beta^2}{\alpha^2\beta^2} \\ 4\frac{\beta^2}{\alpha^2}(\alpha^2-\beta^2) & -\frac{3\alpha^2-2\beta^2}{\alpha^2} \end{bmatrix}$	$\begin{bmatrix} 1 & k_x^2 \\ 2\beta^2 & -\omega^2+2\beta^2 k_x^2 \end{bmatrix}$
$\begin{bmatrix} \frac{1}{\rho}\tau_{zz} \\ \partial_z u \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\alpha^2} & -2\frac{\alpha^2-\beta^2}{\alpha^2} \\ 0 & \frac{1}{\beta^2} \end{bmatrix}$	$\begin{bmatrix} \frac{\alpha^2-2\beta^2}{\alpha^2} & 4\frac{\beta^2}{\alpha^2}(\alpha^2-\beta^2) \\ -\frac{\alpha^2-\beta^2}{\alpha^2\beta^2} & -\frac{3\alpha^2-2\beta^2}{\alpha^2} \end{bmatrix}$	$\begin{bmatrix} k_x^2 & 1 \\ -\omega^2+2\beta^2 k_x^2 & 2\beta^2 \end{bmatrix}$
$\begin{bmatrix} \tau_{xx} \\ \tau_{zz} \end{bmatrix}$	$\begin{bmatrix} \frac{\alpha^2+2\beta^2}{2\alpha^2\beta^2} & -\frac{\alpha^2-2\beta^2}{2\alpha^2\beta^2} \\ -\frac{\alpha^2}{2\alpha^2\beta^2} & \frac{\alpha^2}{2\alpha^2\beta^2} \end{bmatrix}$	$\begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \omega^2\frac{\alpha^2-2\beta^2}{2\alpha^2\beta^2}+k_x^2 & 1 \\ \frac{\omega^2}{2\beta^2}-k_x^2 & -1 \end{bmatrix}$
$\begin{bmatrix} \tau_{xx}+\tau_{zz} \\ \tau_{xx}-\tau_{zz} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\alpha^2} & 0 \\ \frac{1}{\alpha^2} & \frac{1}{\beta^2} \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -2 & -1 \end{bmatrix}$	$\begin{bmatrix} \omega^2\frac{\alpha^2-\beta^2}{\alpha^2\beta^2} & 0 \\ -\frac{\omega^2}{\alpha^2}+2k_x^2 & -\omega^2+2\beta^2 k_x^2 \end{bmatrix}$

TABLE 6.1. The coefficient matrices for four of the mixed variables are given in the table. In the first column are the state variables. The second and third columns are the coefficient matrices of the wave equation given at the top. The last column is the transformation operator which converts P and S potentials into the given variable.

In equation (6.8), and in the following discussion it is assumed that the density ρ and the compressional and shear velocities α and β are constants. The structure of equation (6.8) allows us to construct a second order differential equation for r

$$\frac{\partial^2}{\partial z^2} r = AB\, r$$

In a similar manner, a second-order partial differential equation in the variables $[\partial_x w$ and $(1/\rho)\tau_{xz}]$ can be derived. Another possibility is to begin again with the five first-order equations of motion and eliminate all variables except the normal stresses τ_{xx} and τ_{zz}. This results in another second-order partial differential equation describing elastic wave propagation. The property common to all of these second-order wave equations is their form. Each can be written in the form

$$\partial_{zz} q \quad = \quad (M_1 \partial_{tt} + M_2 \partial_{xx}) q \tag{6.10}$$

The interesting thing about equation (6.10) is that it contains no cross-derivatives (∂_{xz} terms). Other than the fact that M_1 and M_2 are 2 by 2 matrices, and q is a vector with two elements, the form of (6.10) is identical to the scalar wave equation. In Table 6.1, we give the matrices M_1 and M_2 and the state variables q for four differential equations that have the form (6.10). In addition we include the matrix operator Q which can be used to diagonalize equation (6.10). By making the change of variables

$$\tilde{q} \quad = \quad Q^{-1} q \tag{6.11}$$

equation (6.9) becomes a decoupled system (i.e. transform it to P and S variables). Note that both Q and Q^{-1} have an x-dependence of the form ∂_{xx} (or equivalently k_x^2). This means that both the forward and inverse transformations can be implemented with a tri-diagonal operator in the (x,ω) domain.

The fact that equation (6.10) looks like the scalar wave equation is what makes the derivation of the corresponding one-way wave equations so simple. For convenience, let us Fourier transform (6.10) over t to obtain

$$\partial_{zz} q = (-\omega^2 M_1 + M_2 k_x^2) q \tag{6.12}$$

One-way extrapolators can now be formed by taking the square root of equation (6.12).

$$\frac{\partial}{\partial z} q = \pm S_n\, q \tag{6.13}$$

where S_n is the nth approximate to the exact square root $\sqrt{-\omega^2 M_1 + M_2 \partial_{xx}}$. The square root approximations can be generated by a slight generalization of Muir's recurrence relation

$$S_n = -i\omega M_1^{1/2} + (-i\omega M_1^{1/2} + S_{n-1})^{-1} M_2 \partial_{xx}, \qquad S_0 = -i\omega M_1^{1/2} \qquad (6.14)$$

Operating on both sides of equation (6.14) will confirm that S_n converges to the exact square root as $n \to \infty$.

The dispersion relations for the "mixed" set of variables can be determined by finding the eigenvalues of the operators S_n. In Figure 6.1, the dispersion relations for the state variables $[\partial_x u\ , (1/\rho)\tau_{zz}]^T$ are shown. The results are similar for the other state variables. By graphical comparison the dispersion appear to be identical to the corresponding scalar equations.

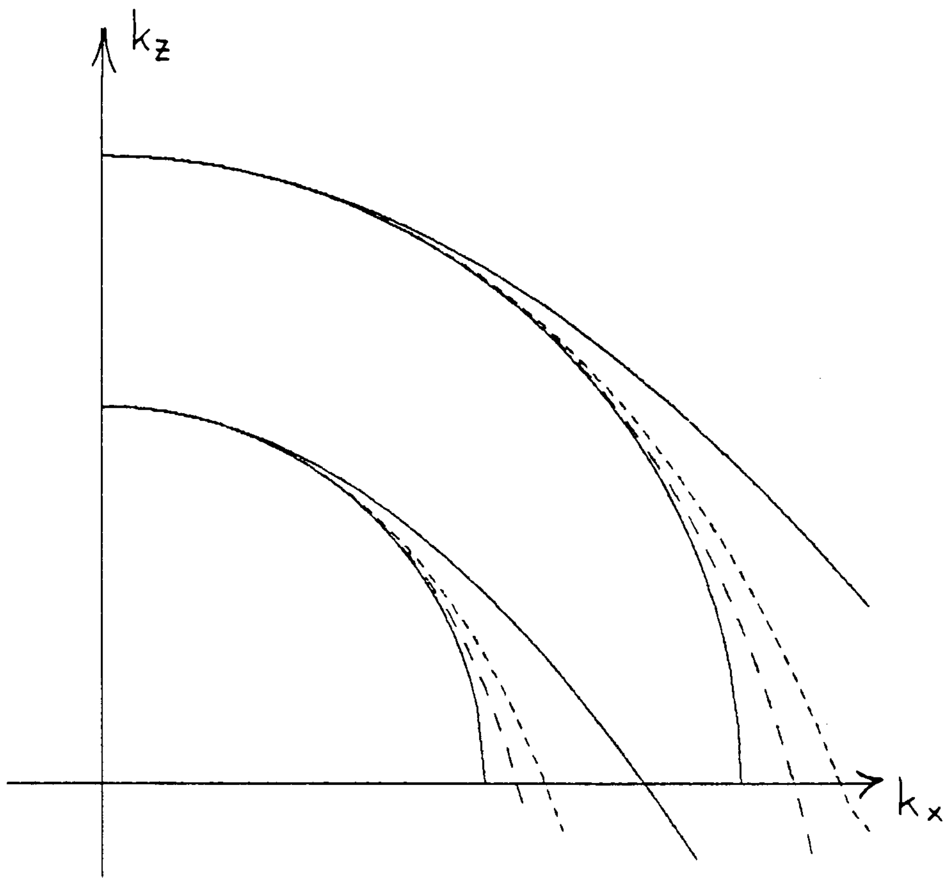

FIG. 6.1. Dispersion curves for one way elastic wave equations in the variables $[\partial_x u, (1/\rho)\tau_{zz}]^T$. The solid line is the first order approximation; the short-dash line is the second order; and the long dash line is the third order.

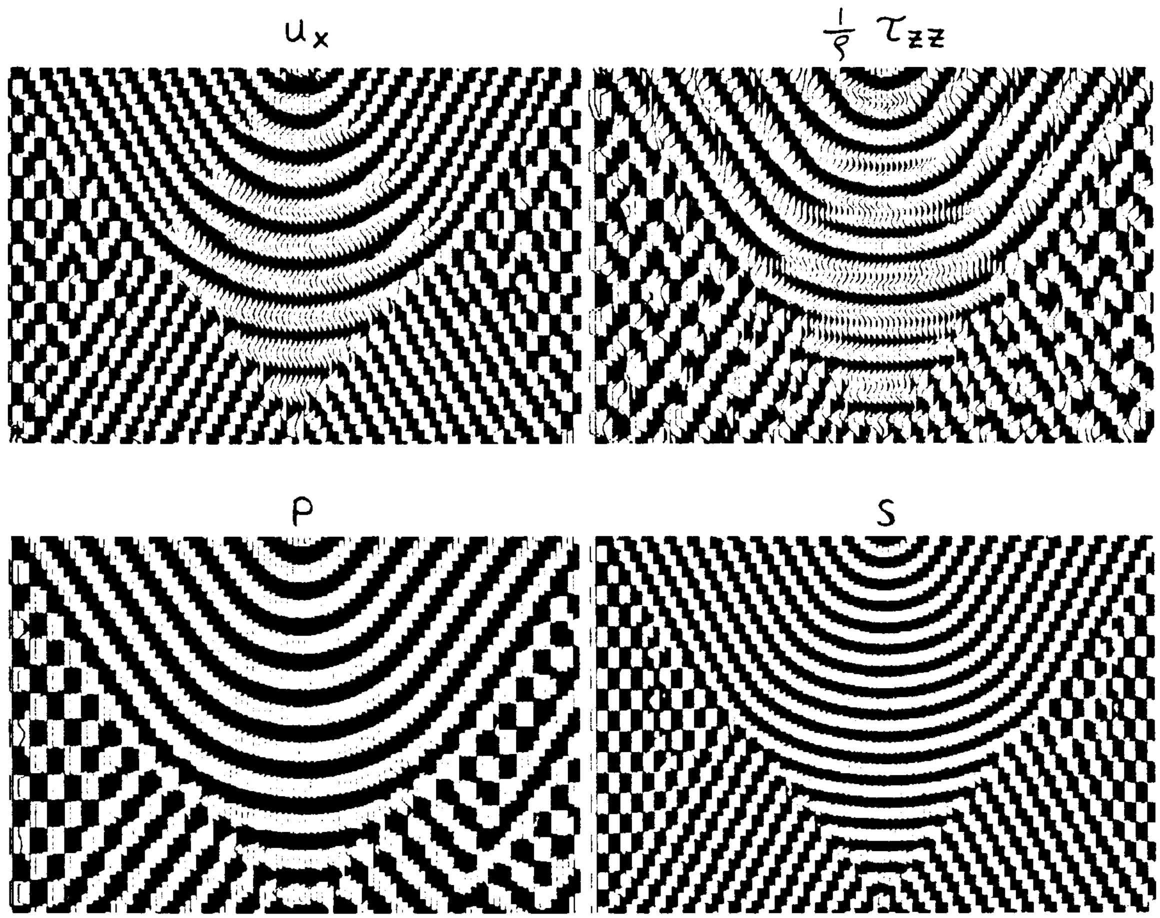

FIG. 6.2. An example of extrapolation with the first set of mixed variables listed in Table 6.1. The initial conditions were constructed from a point source that is equal strength in both P and S. The Q transformation given in Table 6.1 was then applied to provide initial conditions for the mixed variables. The top two panels show the extrapolation using the mixed variables. The bottom two panels show the results of transforming the solution to P and S. The P and S results are virtually identical with those obtained by scalar extrapolation.

The one-way elastic wave equations that we have derived in this section can be useful to us only if they are stable. Moreover, if we are interested in accurate modeling of elastic wave propagation, we would like the equations to be unconditionally stable and to mimic the proper reflection and refraction effects for elastic waves at internal interfaces. We have done some numerical experiments that show that we can find one-way equations in any of the variables mentioned in Table 6.1 that are stable for the constant coefficient case. Figure 6.2 shows the results of one such calculation. In this figure, the the state variables $q = [\partial_x u, (1/\rho)\tau_{zz}]^T$ were used. The initial conditions at $z = 0$ corresponded to the analytic solution of the full elastic wave equation for a point source of equal strength in P and S placed above the top of the plot. An implicit Crank-Nicolson method was used in the finite differencing. The top two plots show the solution in the variables $\partial_x u$ and $(1/\rho)\tau_{zz}$. The bottom two plots show the solution transformed to P and S. The main thing to note is that the solution is stable. Figure 6.4 shows similar plots, but this time a vertical velocity interface was placed in the medium slightly to the right of the point source. The discrete L_2–norm of the solution grew exponentially in z, indicating that the equation is unstable. This is apparent in the figure.

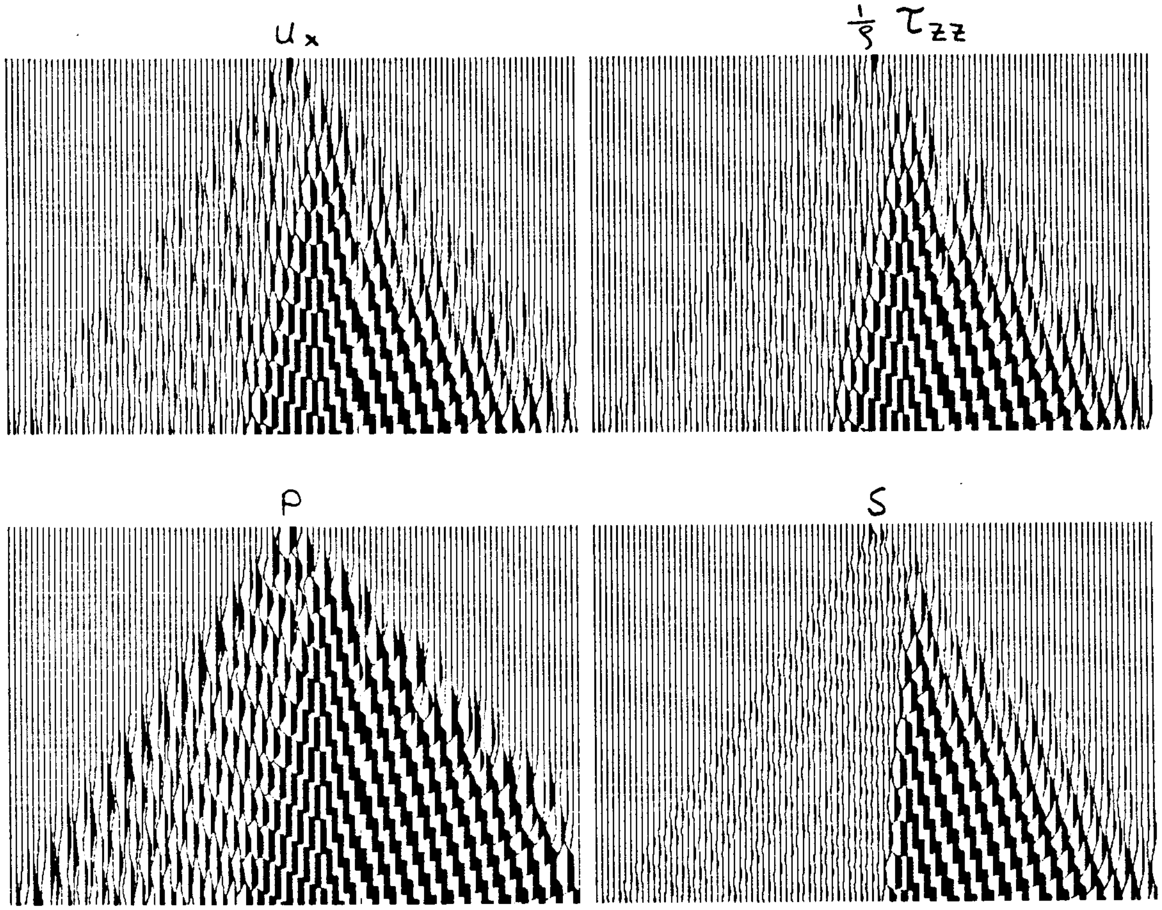

FIG. 6.3. This figure shows an example of an instability in the the mixed variable extrapolation. The top panel show the extrapolation using the mixed variables listed first in Table 6.1. A vertical interface with a 30% velocity contrast is located just to the right of the point source. The waves transmitted and reflected by the interface are growing exponentially. The bottom panel shows the solution transformed to P and S waves.

We now believe that all such equations are probably unstable for the case of strong lateral velocity variation. Determining unconditionally stable approximations to differential equations appears to be closely related to the idea of finding difference approximations that produce reasonable reflection and transmission effects at internal interfaces. If the reflection and transmission coefficients are of reasonable size, then the solution will not blow up when energy strikes the interfaces. As long as we are going to require that the method yield reasonable reflection and transmission coefficients, we might want to require that these coefficients be good approximations to the reflection and transmission coefficients for the full elastic problem. This seems to be where the biggest problem is with all the variable sets given in Table 6.1. It appears that it is not possible to write down conditions that are equivalent to the requirements that all stresses be continuous at an interface using only the two variables in each set.

The difference approximation used to produce the plots in Figure 6.3 was one which implicitly specified the conditions that q and $\partial_x q$ be continuous across the vertical velocity interface. It is fairly simple to calculate explicitly what the reflection and transmission coefficients are for these interface conditions. It is conventional to calculate the ratio of the reflected and refracted P and S wave amplitudes to the incident P or S wave amplitude. So we need first to transform from the variables q to the potential variables P and S. Finding the potential variables amounts to diagonalizing the wave equation (6.12). Define a diagonal matrix Λ_m by

$$\Lambda_m = Q^{-1}MQ \tag{6.15}$$

Then if we define new variables $\tilde{q}$ by

$$\tilde{q} = Q^{-1}q, \tag{6.16}$$

equation (6.12) becomes

$$\frac{\partial^2 \tilde{q}}{\partial z^2} - \Lambda_m \tilde{q} = 0. \tag{6.17}$$

We will only calculate the reflection and transmission coefficients for an incident P-wave. A plane incident P-wave has the form

$$\tilde{q}_o = [A \exp i(k_z z + k_{\alpha_1} x), 0]^T,$$

where k_{α_1} is the horizontal wave-number of the incident P-wave in the first medium. The reflected wave will have the form

$$\tilde{q}_R = [R_p \exp i(k_z z - k_{\alpha_1} x), R_s \exp i(k_z z - k_{\beta_1} x)]^T,$$

and the refracted wave will have the form

$$\tilde{q}_T = [T_p \exp i(k_z z + k_{\alpha_2} x),\ T_s \exp i(k_z z + k_{\beta_2} x)]^T .$$

Here k_{α_2} is the horizontal wavenumber of a P-wave in the second medium, and k_{β_1} and k_{β_2} are the horizontal wavenumbers of an S-wave in the first and second medium, respectively. The interface conditions that q and $\partial_x q$ be continuous across an interface at $x = 0$ can be written in terms of Q^{-1} and $\tilde{q}_o$, $\tilde{q}_R$ and $\tilde{q}_T$:

$$Q^{-1}\tilde{q}_o + Q^{-1}\tilde{q}_R = Q^{-1}\tilde{q}_T$$

$$\text{and} \quad Q^{-1}\partial_x\tilde{q}_o + Q^{-1}\partial_x\tilde{q}_R = Q^{-1}\partial_x\tilde{q}_T . \tag{6.18}$$

Let us take as an example the variables $q = [\partial_x w, (1/\rho)\tau_{xz}]^T$. Q^{-1} is given by

$$Q^{-1} = \frac{1}{\omega^2}\begin{pmatrix} -\omega^2 - 2\beta_2\partial_{xx} & \partial_{xx} \\ -2\beta_2 & 1 \end{pmatrix}$$

Letting $a = [A, 0]^T$, $r = [R_p, R_s]^T$, and $\tau = [T_p, T_s]^T$, equations (6.18) become

$$a + r = Q_o^{-1} Q_1 \tau$$

$$\text{and} \quad a - r = \Lambda_o^{-1} Q_o^{-1} Q_1 \Lambda_1 \tau \tag{6.19}$$

where

$$Q_o = \begin{pmatrix} -\omega^2 + 2\beta^2 k_{\alpha_1}^2 & -k_{\alpha_1}^2 \\ -2\beta^2 & 1 \end{pmatrix} \quad Q_1 = \begin{pmatrix} -\omega^2 + 2\beta^2 k_{\alpha_2}^2 & -k_{\alpha_2}^2 \\ -2\beta^2 & 1 \end{pmatrix}$$

$$\Lambda_o = \begin{pmatrix} ik_{\alpha_1} & 0 \\ 0 & ik_{\beta_1} \end{pmatrix} \quad \text{and} \quad \Lambda_1 = \begin{pmatrix} ik_{\alpha_2} & 0 \\ 0 & ik_{\beta_2} \end{pmatrix}$$

Equations (6.19) can be solved explicitly for r and τ as a function of the incident angle $\vartheta = \cos^{-1}(\alpha_1 k_{\alpha_1}/\omega)$ and a. This was done numerically and the results are shown in Figure 6.4. The problem with these reflection and transmission coefficients is the singularity; the coefficients become very large for certain incident angles. This would seem to insure instability. We tried all of the other sets of variables listed in the table and several other boundary conditions of the type

$$\Gamma_1 \partial_x q_1 = \Gamma_2 \partial_x q_2$$

where Γ_1 and Γ_2 are 2 by 2 matrices, but all of the ones tried exhibited the same features as the plot in Figure 6.4. We concluded that the instability is probably

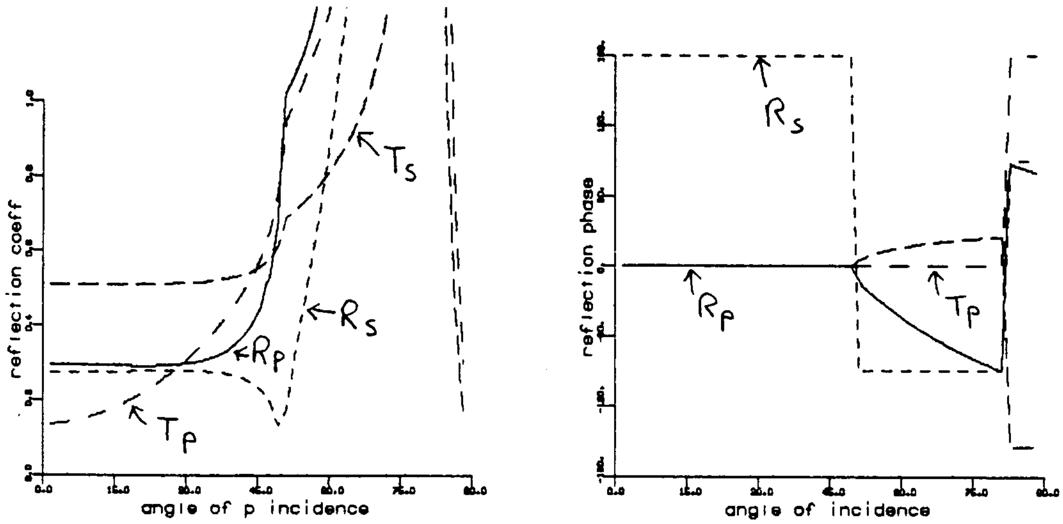

FIG. 6.4. The figure shows the apparent reflection and transmission coefficients, for the first set of mixed variables listed in Table 6.1. The incident wave is a P wave. The singularity in both the reflection and transmission coefficients appears to cause the instability shown in Figure 6.3.

unavoidable with these differential equations.

Conclusions

We have not been able to conclusively select a set of variables for elastic wave extrapolation. Each of the three types discussed fail to satisfy all of the requirements of an ideal operator. We feel, however, that for modeling purposes the displacements are the best choice. The present lack of higher order approximations limits the angular accuracy to a cone about the extrapolation direction. However, since the internal boundary conditions are satisfied the solutions should be correct within the cone.