FOURIER TRANSFORMS OF FUNCTIONS WITH ASYMPTOTES

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My basic inclination is to try to Fourier transform every function with a straight FFT algorithm, and hang the consequences. Actually, since the FFT algorithm assumes a periodic function, and since most of the functions we meet are anything but periodic, some modifications of the basic FFT algorithm really ought to be employed. This is especially true for functions which require interpolation in the frequency domain.

The modifications themselves are pretty simple. My chief problem is that I never write them down, so I keep having to rederive them. Maybe if I write them down here, and take a vow never to wander more than 50 feet from the nearest SEP-24, all my problems will go away.

Suppose $f(t)$ is a time function with asymptotes

$$f(t) = f_- e^{\epsilon t} \qquad t < 0$$

$$= f_+ e^{-\epsilon(t-t_f)} \qquad t > t_f$$

where ϵ is some incredibly small number, and f_- and f_+ are known constants. Then the Fourier transform $\bar{f}(\omega)$ of $f(t)$ will be

$$\bar{f}(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} f(t)$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \left[f_- \int_{-\infty}^{\infty} dt\ e^{-t(i\omega-\epsilon)} + \int_0^{t_f} dt\ e^{-i\omega t}\ f(t) \right.$$

$$\left. + f_+ \int_{t_f}^{\infty} dt\ e^{-i\omega t}\ e^{-\epsilon(t-t_f)} \right] \tag{1}$$

Performing the first and last integral, this becomes

$$\bar{f}(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \left[f_- \frac{i}{\omega + i\epsilon} + \int_0^{t_f} dt\ e^{-i\omega t}\ f(t) - \frac{f_+ i e^{-i\omega t_f}}{\omega - i\epsilon} \right]$$

or, using

$$\frac{1}{\omega \pm i\epsilon} = \frac{P}{\omega} \pm i\pi\delta(\omega) \qquad \text{(P means principal value)}$$

we get

$$\bar{f}(\omega) = \frac{1}{(2\pi)^{\frac{1}{2}}} \left[\int_0^{t_f} dt\ e^{-i\omega t}\ f(t) + i\left(f_- - f_+ e^{-i\omega t_f}\right) \frac{P}{\omega} + \left(f_- + f_+ e^{-i\omega t_f}\right) \pi\delta(\omega) \right] \tag{2}$$

We can digitize this expression under the assumption that $f(t)$ is band-limited. Let Δt be an increment in t such that $t_f/\Delta t \equiv NT$. Choose $\Delta\omega = 2\pi/t_f$. Then

$$t \rightarrow t_j = j\Delta t\ , \quad \omega \rightarrow \omega_k = k\Delta\omega$$

Define

$$g_j \equiv f(t_j) \tag{3a}$$

$$\bar{g}_k \equiv \frac{1}{\Delta t} \int_0^{t_f} dt\ e^{-i\omega_k t}\ f(t) \tag{3b}$$

If $\bar{g}_k = 0$ for $k \notin (-NT/2, NT/2)$, then

$$\bar{g}_k = \sum_{j=0}^{NT-1} g_j \, e^{-i2\pi jk/NT} \tag{4a}$$

$$g_j = \frac{1}{NT} \sum_{R=NT/2+1}^{NT/2} \bar{g}_k \, e^{i2\pi jk/NT} \tag{4b}$$

So, if we digitize $\bar{f}$ at intervals $\Delta\omega$

$$\bar{f}(\omega_k) \rightarrow \frac{\Delta t}{(2\pi)^{\frac{1}{2}}} \left[\bar{g}_k + i(f_- - f_+) \frac{NT}{2\pi} \frac{1 - \delta_{ko}}{k} + (f_- + f_+) \frac{NT}{2} \delta_{ko} \right] \tag{5}$$

Suppose, however, that we need to know $\bar{f}(\omega)$ at intervals smaller than $\Delta\omega$. Define $\Delta\omega' = \Delta\omega/m$, and let

$$\bar{f}_m(n) \equiv \bar{f}\left(\frac{n\Delta\omega}{m}\right)$$

$$= \frac{\Delta t}{(2\pi)^{\frac{1}{2}}} \left[\frac{1}{\Delta t} \int dt \; f(t) \, e^{-in\Delta\omega t/m} + \frac{i(1-\delta_{no})}{n} \frac{mNT}{2\pi} \left(f_- - f_+ e^{(-i2\pi n)/m} \right) + \right.$$

$$\left. + \frac{mNT}{2} \delta_{no} \left(f_- + f_+ e^{(-i2\pi n)/m} \right) \right]$$

The integral in this expression can be put in terms of the $\bar{g}_k$. If $n = mK + \ell$, $0 \leq \ell < m$, then

$$\bar{f}_m(n) = \frac{\Delta t}{(2\pi)^{\frac{1}{2}}} \left[\sum_{k'=NT/2+1}^{NT/2} \bar{g}_{k'} \, A(k'-k, \ell) + \frac{i(1-\delta_{no})}{n} \frac{mNT}{2\pi} \left(f_- - f_+ e^{-i2\pi m/n} \right) + \right.$$

$$\left. + \frac{mNT}{2} \delta_{no} \left(f_- + f_+ e^{-i2\pi n/m} \right) \right] \tag{6}$$

where

$$A(k, \ell) = \frac{1}{NT} \frac{e^{-i2\pi \ell/m} - 1}{e^{i2\pi(k-\ell/m)/NT} - 1} \tag{7}$$

Equation (6) would seem to be the proper way to interpolate (in frequency) functions with asymptotes. In practice, one might prefer to substitute for $A(k,\ell)$ an approximate expression which is nonzero only for small k.