INVERSION OF SEISMIC DATA IN A LATERALLY HETEROGENEOUS MEDIUM

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It would be nice[1] to have an algorithm for full inversion of seismic data, multiples and all, which does not depend on starting with a "good" estimate of elastic (acoustic) parameters. The Gelfand-Levitan method offers a sort of mindless inversion scheme for layered media, and if it could be applied to the general seismic problem I feel it would be worth doing. Unfortunately, this method seems to require a "potential" which is diagonal in at least one direction and, further, which is independent of frequency. While the wave equation is easily modified to meet one requirement or the other, it does not appear possible in general to meet both. Therefore we reluctantly conclude that the Gelfand-Levitan inverse method is not, at least in its pristine form, applicable to the general inverse seismic problem.

This does not mean, however, that there is no solution to the inverse seismic problem; it just means that we must use another method.[2] We develop below an inversion scheme directly applicable to the acoustic wave equation and general enough, we hope, to be easily extendable to the elastic wave equation.

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[1]A question to keep in mind while reading this article is how much would a magic inversion process be worth, measured in units of increased computer and data acquisition costs. If the answer isn't "plenty," the reader can stop here.

[2]Lest the reader think this is a new idea, it is not. Jost and Kohn (September 1952, Phys. Rev., vol. 87, no. 6, p. 977) predate Gelfand-Levitan. Razavy (November 1975, J. Acoust. Soc. Am., vol. 58, no. 5, p. 956) develops an iterative inverse for the 1-D constant density case. Weglein, Boyse and Anderson (submitted to Geophysics, 1979) present the 3-D constant density in-

We begin with the scalar wave equation

$$\left(\nabla \cdot \frac{1}{\rho} \nabla + \frac{\omega^2}{K}\right) \psi = 0 \qquad (1a)$$

Comparing it with the constant parameter equation

$$\left(\frac{1}{\rho_0} \nabla^2 + \frac{\omega^2}{K_0}\right) \psi_0 = 0 \qquad (1b)$$

gives us a non-local, frequency-dependent potential operator

$$V(\omega,x) = \omega^2 \frac{a(x)}{K_0} + \nabla \cdot \frac{b(x)}{\rho_0} \nabla \qquad (2)$$

where

$$a = \frac{K_0}{K} - 1 \quad \text{and} \quad b = \frac{\rho_0}{\rho} - 1 ,$$

the same as Clayton has used for the simple Born approximation.

The inverse problem, in which one attempts to find V from a limited set of data, would be hopeless if V were an arbitrary operator. V is hardly arbitrary, however. It has almost as much structure (half as much, to be precise) as a Schroedinger potential, and there is no reason to suppose that given a reasonable amount of data we can't solve for it. In fact we know it can be solved to first (Born) approximation.

We will attempt a solution for V of the Lippmann-Schwinger equation

$$G = G_0 + G_0VG = G_0 + GVG_0 \qquad (3)$$

where G_0 is the known Green's function for the unperturbed wave operator (1b)

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verse problem as a system of coupled integral equations which are not unlike those developed in this paper. Indeed, this paper could be viewed as an extension of the work of Weglein, et al.

and G is the Green's function for the variable wave operator (1a), unknown in general but measured for sources and receivers on the earth's surface.

It is convenient to bring from the quantum scattering closet the scattering operator T, defined by[3]

$$T(\omega) = V + VGV \tag{4}$$

It will be seen shortly that T is actually the measured quantity (partly measured, anyway) in the seismic experiment. Postmultiplying (4) by G_0 and applying (3) yields the relation

$$VG = TG_0 \tag{5}$$

which may be used to modify the Lippmann-Schwinger equation to

$$G = G_0 + G_0TG_0 \tag{6}$$

and to turn (4) into a Lippmann-Schwinger equation for T:

$$T = V + TG_0V \tag{7}$$

Equations (6) and (7) will form the nucleus of our inversion scheme.

We take our measured data field to be

$$D(\omega,x_r|x_s) = \langle x_r,z_r=0|(G-G_0)|x_s,z_s=0\rangle$$

or equivalently, if we Fourier transform $x_r \to p_r$ and $x_s \to p_s$,

$$\bar{D}(\omega,p_r|p_s) = \langle p_r,o|(G - G_0)|p_s,o\rangle$$

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[3]For a very lucid description of the T operator, see Taylor, 1972, ***Scattering Theory***: Wiley, p. 134.

At this point we substitute for $G - G_o$ the exact expression (6), giving

$$\bar{D}(\omega,p_r|p_s) = \langle p_r,o|G_oTG_o|p_s,o\rangle \tag{8}$$

Equation (8) can be put in the form of an integral equation by inserting complete sets of states between G_o and T and between T and the second G_o:

$$\bar{D}(\omega,p_r|p_s) = \int d\vec{x} \int d\vec{x}' \langle p_r,o|G_o|\vec{x}\rangle\langle\vec{x}|T|\vec{x}'\rangle\langle\vec{x}'|G_o|p_s,o\rangle \tag{9}$$

making use of the explicit forms for G_o:

$$\langle\vec{x}'|G_o|p_s,o\rangle = \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{i\rho_o}{2q_s} e^{i(p_s x' + q_s z')} \qquad (z' > o)$$

$$\langle p_r,o|G_o|\vec{x}\rangle = \frac{1}{(2\pi)^{\frac{1}{2}}}\frac{i\rho_o}{2q_r} e^{-i(p_r x - q_r z)} \qquad (z > o)$$

with

$$q_s = \left(\frac{\omega^2}{v_o^2} - p_s^2\right)^{\frac{1}{2}}$$

$$q_r = \left(\frac{\omega^2}{v_o^2} - p_r^2\right)^{\frac{1}{2}} \tag{10}$$

$$v_o = \left(\frac{K_o}{\rho_o}\right)^{\frac{1}{2}}$$

Equation (9) becomes (since $\langle\vec{x}|T|\vec{x}'\rangle = o$ if z or $z' < 0$)

$$\bar{D}(\omega,p_r|p_s) = \frac{-\rho_o^2}{8\pi q_s q_r} \int d\vec{x} \int d\vec{x}' \, e^{-i(p_r x - q_r z)} \langle\vec{x}|T|\vec{x}'\rangle \, e^{i(p_s x' + q_s z')}$$

The integrals in this equation are obviously just Fourier transforms, and we can write

$$\bar{D}(\omega,p_r|p_s) = \frac{-2\pi}{4q_sq_r}\rho_o^2 \langle p_r,-q_r|T(\omega)|p_s,q_s\rangle \tag{11}$$

All that math has boiled down to a simple statement: the measured seismic field is just the T matrix. More precisely, it is the "on shell" T matrix, since, according to equation (10), p_r and q_r (and p_s and q_s), for a given ω, are confined to the spherical shell $p_r^2 + q_r^2 = \omega^2/v_o^2$. [Also, $p_s^2 + q_s^2 = \omega/v_o^2$].

If T were known everywhere, the inverse problem would be solved (in principle, anyway), since according to equation (6), if we know T we can compute G, and if we know G we can compute the real wave operator, potential and all. We don't know T everywhere, so if T were an arbitrary operator we would be sunk. But equation (7) gives T a structure which with luck will, together with equation (11), define it uniquely. Substituting (7) into (11) we obtain

$$\bar{D}(\omega,p_r,p_s) = \frac{-2\pi\rho_o^2}{4q_sq_r}\langle p_r,-q_r|(1 + TG_o)V|p_s,q_s\rangle \tag{12}$$

Equation (12) could be discretized into a set of matrix equations. The number of equations would be $N\omega \times Np_r \times Np_s$. Since the number of unknown T values would be $Np_r \times Np_s \times Nq_r \times Nq_s$, and the number of V_1 and V_2 values together would be $2 \times Np \times Nq$, it's pretty clear there is not enough information in (12) to directly determine T and V. However, given T, there are more than enough equations in (12) to find V. This suggests the following iterative scheme.

From a first estimate of T, we use (12) to determine a first estimate of V (if the first estimate of T is zero, the first estimate of V amounts to the Born approximation). We will then use equation (7) [which is just the off-shell extension of (12)] to form a new estimate of T. The new estimate of T is returned to (12) to obtain a new estimate of V and so on. We continue cycling through (12) and (7) until we converge to an acceptable solution or our computer dies of old age.

It is possible to imagine more than one iterative scheme which cycles through (12) and (7) as described above. The details of a couple of the more straightforward are as follows.

Scheme 1

Imagine a sequence of estimates $\{(T_1,V_1),(T_2,V_2),...\}$ which hopefully will converge to the desired result (T, V). T_1 will be set to zero. Given T_m, V_m will be found by inverting the equation

$$\bar{D}(\omega,p_r|p_s) = \frac{-2\pi\rho_o^2}{4q_s q_r}\langle p_r,-q_r|(1 + T_m G_o)V_m|p_s,q_s\rangle \qquad (13)$$

Given V_m, T_{m+1} can be found from

$$\langle p_r,-q_r|T_{m+1}|p,q\rangle = \langle p_r,-q_r|(1 + T_m G_o)V_m|p,q\rangle \qquad (14)$$

p_r and q_r may be left "on shell" in equation (14), since only their on-shell values will be needed in (13). However, the right-hand variables p and q must be allowed "off shell."

To actually compute equations (13) and (14), we would expand them out as integral equations. For example, we may put a complete set of states just ahead of V_m in (14) to obtain

$$\langle p_r,-q_r|T_{m+1}|p,q\rangle = \int\int dp'dq'\langle p_r,-q_r|(1 + T_m G_o)|p',q'\rangle *$$

$$*\langle p',q'|V_m|p,q\rangle$$

We may make use of the fact that $|p',q'\rangle$ is an eigenvector of G_o

$$G_o|p',q'\rangle = \frac{-\rho_o}{\left(\frac{\omega^2}{V_o^2} + i\epsilon - p'^2 - q'^2\right)}|p',q'\rangle$$

to write

$$\langle p_r, -q_r | T_{m+1} | p, q\rangle = \int\int dp'dq' \left[\delta(p_r - p')\delta(q_r + q') - \right.$$

$$\left. -\rho_o \frac{\langle p_r, -q_r | T_m | p', q'\rangle}{\frac{\omega^2}{v_o^2} + i\epsilon - p'^2 - q'^2} \right] * \langle p', q' | V_m | p, q\rangle \qquad (15)$$

From equation (2) for V we have as a form for the V matrix

$$\langle p', q' | V_m | p, q\rangle = \frac{1}{2\pi} \left[\omega^2 \frac{a_m(p'-p, q'-q)}{K_o} - \right.$$

$$\left. - (pp'+qq') \frac{b_m(p'-p, q'-q)}{\rho_o} \right] \qquad (15a)$$

where a and b are the Fourier transforms of the potential components $a(\vec{x})$ and $b(\vec{x})$.

We can express the T-matrix elements in the functional form

$$\langle p_r, \frac{-\omega}{v_o^2} \left(1 - \frac{p_r^2 v_o^2}{\omega^2}\right)^{\frac{1}{2}} | T_m(\omega) | p', q'\rangle \equiv T_m(\omega, p_r | p', q') \qquad (15b)$$

making use of the fact that q_r was actually a function of p_r and ω. Putting these two expressions into (15) yields

$$T_{m+1}(\omega, p_r | p, q) = \int dp' \int dq'$$

$$\left[\delta(p_r - p')\delta(q_r + q') - \rho_o \frac{T_m(\omega, p_r | p', q')}{\frac{\omega^2}{v_o^2} + i\epsilon - p'^2 - q'^2} \right] *$$

$$* \frac{1}{2\pi}\left[\omega^2 \frac{a_m(p' - p, q' - q)}{K_o} - (pp' + qq') \frac{b_m(p' - p, q' - q)}{\rho_o}\right]$$

or, defining $p'' = p' - p$, $q'' = q' - q$,

$$T_{m+1}(\omega, p_r | p, q) = \frac{1}{2\pi} \int dp'' \int dq''$$

$$\left[\delta(p_r - p - p'')\,\delta(q_r + q + q'') - \rho_o \frac{T_m(\omega, p_r | p + p'', q + q'')}{\frac{\omega^2}{v_o} + i\epsilon - (p + p'')^2 - (q + q'')^2}\right] *$$

$$* \left\{\omega^2 \frac{a_m(p'', q'')}{K_o} - [(p + p'')p + (q + q'')q] \frac{b_m(p'', q'')}{\rho_o}\right\} \tag{16}$$

This equation may look a little messy, but its evaluation should be quite straightforward.

Since equation (13) is just the "on shell" restriction of (14), its expression as an integral equation will be almost identical. We write

$$\bar{D}(\omega, p_r, p_s) = \frac{-\rho_o^2}{4q_r q_s} \int\int dp''dq'' \left[\delta(p_r - p_s - p'')\right.$$

$$\left.\delta(q_r + q_s + q'') - \rho_o \frac{T_m(\omega, p_r | p_s + p'', q_s + q'')}{\frac{\omega^2}{v_o^2} + i\epsilon - (p_s + p'')^2 - (q_s + q'')^2}\right] *$$

$$* \left\{\omega^2 \frac{a_m(p'', q'')}{K_o} - [(p_s + p'')p_s + (q_s + q'')\,q_s] \frac{b_m(p'', q'')}{\rho_o}\right\} \tag{17}$$

Viewed as a matrix equation to invert, we could express (17) as

$$\bar{D}(\omega, p_r, p_s) = -\sum_{\alpha=1}^{2} \int\int dp''dq''\, M_m(\omega, p_r, p_s | p'', q'', \alpha)\, \bar{V}_\alpha^{(m)}(p'', q'') \tag{18}$$

where the operator M has elements

$$M_m(\omega, p_r, p_s | p'', q'', 1) = -\rho_0^2 \frac{\omega^2}{4 q_r q_s} *$$

$$\left[\delta(p_r - p_s - p'')\, \delta(q_r + q_s + q'') - \rho_0 \frac{T_m(\omega, p_r | p_s + p'', q_s + q'')}{\frac{\omega^2}{v_0^2} + i\epsilon - (p_s + p'')^2 - (q_s + q'')^2} \right]$$

and

$$M_m(\omega, p_r, p_s | p\ ,\ q\ , 2) = -\frac{p_s(p_s + p'') + q_s(q_s + q'')}{\omega^2} M_m(\omega, p_r, p_s | p'', q'', 1)$$

and the vector $\bar{V}$ is just

$$\bar{V}_1^{(m)} = \frac{a_m}{K_0}, \quad \bar{V}_2^{(m)} = \frac{b_m}{\rho_0}$$

It is pretty clear that for any reasonable discretization, we will have more equations than unknowns, suggesting a least-squares (or perhaps some other exotic norm) inversion for V. Of course, M_m is a rather large matrix, and no guarantee of an inverse has been presented here. Nevertheless we have here at least a formal, if somewhat pedestrian, solution to the inverse problem.

Scheme 2

It might possibly be desirable to have an iterative scheme which does not require the inversion of a $10^6 \times 10^6$ matrix at every step. One way (suggested by Razavy, 1975, for a constant density layered medium) would be to expand V and T as a power series in the gross amplitude of the data. That is, we write

$$\bar{D}(\omega, p_r | p_s) = a\bar{D}_0\,(\omega, p_r | p_s)$$

$$T = \sum_{m=1}^{\infty} a^m T_m$$

$$V = \sum_{m=1}^{\infty} a^m V_m$$

Equating powers of a in equation (7) we obtain

$$T_m = V_m + \sum_{m'<m} T_{m'} G_o V_{m-m'}$$

$$= V_m + A_m \tag{19a}$$

with

$$A_m = \sum_{m<m} T_m \, G_o \, V_{m-m'} \tag{19b}$$

Doing the same with equation (12), we get for $m = 1$

$$\overline{D}_o(\omega, p_r | p_s) = -\frac{2\pi\rho_o^2}{4q_r q_s} \langle p_r, -q_r \, | V_1 | p_s, q_s \rangle \tag{19c}$$

and for $m > 1$

$$\langle p_r, -q_r | V_m | p_s, q_s \rangle = -\langle p_r, -q_r | A_m | p_s, q_s \rangle \tag{19d}$$

These equations may be expressed in integral form, just as was done for Scheme 1. With the definitions

$$\langle p, q | V_m(\omega) | p', q' \rangle = \frac{1}{2\pi} \left[\omega^2 \frac{a_m}{K_o} (p - p', q - q') \right.$$

$$\left. - (pp' + qq') \frac{b_m}{\rho_o} (p - p', q - q') \right]$$

and

$$T_m(\omega,p_r|p,q) = \langle p_r, -\left(\frac{\omega^2}{v_o^2} - p_r^2\right)^{\frac{1}{2}} |T_m(\omega)|p,q\rangle$$

we can express equations (19a) - (19d) in the following form; equation (19c) becomes

$$\frac{-\rho_o^2}{4q_r q_s}\left[\omega^2 \frac{a_1(p_1 - p_s,-q_r-q_s)}{K_o} - (p_r p_s - q_r q_s) \cdot \frac{b_1(p_r - p_s,-q_r-q_s)}{\rho_o}\right] = \bar{D}_o(\omega,p_r|p_s) \qquad (20a)$$

which is just the Born approximation. Since A_1 is zero, we have for T_1

$$T_1(\omega,p_r|p,q) = \frac{1}{2\pi}\left[\omega^2 \frac{a_1(p_r-p,-q_r-q)}{K_o} - (p_r p - q_r q) \cdot \frac{b_1(p_r-p,-q_r-q)}{\rho_o}\right] \qquad (20b)$$

Given $V_{m'}$, $T_{m'}$, for $m' < m$ we have for A_m [using (19b)]

$$A_m(\omega,p_r|p,q) = -\frac{\rho_o}{(2\pi)} \sum_{m'<m} \int dp' \int dq' \frac{T_{m'}(\omega,p_r|p + p',q + q')}{\frac{\omega^2}{v_o} + i\epsilon - (p + p')^2 - (q + q')^2} *$$

$$* \left[\omega^2 \frac{a_{m-m'}(p',q')}{K_o} - [p(p + p') + q(q + q')] \frac{b_{m-m'}(p',q')}{\rho_o}\right] \qquad (20c)$$

Then for V_m [using (19d)]

$$\omega^2 \frac{a_m(p_r-p_s,-q_r-q_s)}{K_o} - (p_r p_s - q_r q_s) \cdot \frac{b_m(p_r-p_s,-q_r-q_s)}{\rho_o} =$$

$$-2\pi A_m(\omega,p_r|p_s,q_s) \qquad (20d)$$

and finally for T_m [equation (19a), this time]

$$T_m(\omega,p_r|p,q) = A_m(\omega,p_r|p,q) + \frac{\omega^2 a_m(p_r-p,-q_r-q)}{2\pi K_o} -$$

$$- (p_r p - q_r q) \cdot \frac{b_m(p_r-p,-q_r-q)}{2\pi \rho_o} \qquad (20e)$$

Equation (20a) actually overdetermines a_1 and b_1, so in practice one would use this equation in a least-squares (or perhaps somewhat more exotic) sense. In equation (20b) we would find that not all the T_1 values are independent. If the dimension of q is equal to that of ω, then only T_1 for two p_r values need be computed to determine V_2. In fact, two p_r values should be sufficient for all A_m and T_m. Even though they are quite large matrices A_m and T_m are not quite so cumbersome as they could be.

There are some other nice things about Scheme 2. Obviously, we have avoided the matrix inverse (possibly at the expense of more iterations, although even that isn't perfectly clear). Moreover, the potential, once we have it, is a power series in the amplitude factor a. This means we can do an inversion even if the absolute amplitude of the data is unknown, picking, once all the V_m's are known, the amplitude which gives the "best" V.

The worst feature of this method is that to calculate V_m, all the matrices T_m', m' = 1,2...m - 1 are used. Try not to think about storage requirements; it will only depress you. The first and second iterations can be calculated with relatively little storage; for some applications they may be sufficient.

Finally, it is not clear at this point how either scheme reacts to band-limited data. In principle, knowledge of D at all p_r allows calculation of

V clear down to zero frequency, but the practice is likely to be another matter.

Inversion with a Local Potential

The inversion formalism developed above dealt with a potential function with both non-local and frequency-dependent components. It is easy enough to modify the scalar wave equation to produce either a local or a frequency-dependent potential (though it seems we can't have both). We will now redo the inversion formalism for a local potential. Define

$$\eta(\bar{x}) = \rho^{-1/2} \quad ; \quad \phi = \eta\psi$$

We then get the reasonably nice equation for ϕ:

$$\left(\nabla^2 + \frac{\omega^2}{v^2} - \frac{(\nabla^2\eta)}{\eta}\right)\phi = 0 \tag{21}$$

which, when compared to a constant parameter equation,

$$\left(\nabla^2 + \frac{\omega^2}{v_o^2}\right)\phi_o = 0 \tag{22}$$

yields as a potential difference

$$V(\omega,\vec{x}) = \omega^2 V_1(\vec{x}) - V_2(\vec{x}) \tag{23}$$

with

$$V_1(\vec{x}) = \frac{1}{v^2} - \frac{1}{v_o^2} \quad ; \quad V_2(\vec{x}) = \frac{\nabla^2\eta}{\eta} \tag{24}$$

It seems that the quantities to solve for are now velocity and density rather than modulus and density, but that's all right.

We might also have employed the transformation described in "Gilding the Born Approximation" to get an almost local potential and vertical traveltime rather than depth as an independent variable, but we will stick with this one for now.

Even though the potential (23) is a little simpler than (3) (at least until one tries to unravel V_2 for π) we can do nothing now we couldn't do before. However, we shall find some of the matrix algorithms slightly simplified. V matrix elements now have the form

$$\langle p,q|V|p'q'\rangle = \frac{1}{2\pi}\left[\omega^2 V_1(p-p',q-q') - V_2(p-p',q-q')\right]$$

To see how this changes things, look at Scheme 2. Rewriting (20a) - (20d) to fit this potential, we have for $m = 1$

$$\frac{-1}{4q_r q_s}\left[\omega_2 V_1^{(1)}(p_r-p_s,-q_r-q_s) - V_2^{(1)}(p_r-p_s,-q_r-q_s)\right] = \bar{D}_o(\omega,p_r|p_s) \quad (25a)$$

$$T_1(\omega,p_r|p,q) = \frac{1}{2\pi}\left[\omega^2 V_1^{(1)}(p_r-p,-q_r-q) - V_2^{(1)}(p_r-p,-q_r-q)\right] \quad (25b)$$

and for $m > 2$, we first find A_m using

$$A_m(\omega,p_r|p,q) = -\frac{1}{2\pi}\sum_{m'<m}\int dp' \int dq' \frac{T_m'(\omega,p_r|p+p',q+q')}{\frac{\omega^2}{V_o^2} + i\epsilon-(p+p')^2-(q+q')^2} *$$

$$* \left[\omega^2 V_1^{(m-m')}(p',q') - V_2^{(m-m')}(p',q')\right] \quad (25c)$$

then V_m according to

$$\omega^2 V_1^{(m)}(p_r-p_s,-q_r-q_s) - V_2^{(m)}(p_r-p_s,-q_r-q_s) = -2\pi A_m(\omega,p_r|p_s,q_s) \quad (25d)$$

and finally T_m

$$T_m(\omega,p_r|p,q) = A_m(\omega,p_r|p,q) + \frac{\omega^2 V_1^{(m)}(p_r-p_s,-q_r-q_s)}{2\pi} +$$

$$- \frac{V_2^{(m)}(p_r-p,-q_r-q)}{2\pi} \qquad (25e)$$

The substantive change is in equation (25c). Note that now the only p (or q) dependence inside the integral is in the form p + p' (or q + q'); that is, the integrals in (25c) are convolutions.

The Layered Medium

What happens if the potential $V(\omega,x,z)$ is actually independent of x? In this case, we can write

$$\langle p,q|V(\omega)|p',q'\rangle = \frac{1}{(2\pi)^{\frac{1}{2}}} \delta(p-p') \left[\omega^2 V_1(q-q')-V_2(q-q')\right]$$

Moreover D_0, A , and T must also be diagonal in p:

$$\bar{D}_0(\omega,p_r|p_s) = (2\pi)^{\frac{1}{2}} \delta(p_r-p_s)\ \bar{D}_0(\omega,p_r)$$

$$T(\omega,p_r|p,q) = \delta(p_r-p)\ T(\omega,p_r|q)$$

$$A(\omega,p_r|p,q) = \delta(p_r-p)\ A(\omega,p_r|q)$$

(I hope using identical symbols for essentially different functions doesn't confuse anyone.)

So, equations (25) collapse to

$$\bar{D}_0(\omega,p_r) = \frac{-1}{8\pi q_r^2}\left[\omega^2 V_1^{(1)}(-2q_r) - V_2^{(1)}(-2q_r)\right] \qquad (26a)$$

for V_1 and

$$T_1(\omega,p_r|q) = \frac{1}{(2\pi)^{\frac{1}{2}}}\left[\omega^2 V_1^{(1)}(-q_r-q) - V_2^{(1)}(-q_r-q)\right] \qquad (26b)$$

for T_1. For m > 1 we first do

$$A_m(\omega,p_r|q) = -\frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{m'<m} \int dq' \frac{T_{m}'(\omega,p_r|q' + q) * \left[\omega^2 V_1^{(m-m')}(q') - V_2^{(m-m')}(q')\right]}{\frac{\omega^2}{v_o^2} + i\epsilon - \rho_r^2 - (q + q')^2} \qquad (26c)$$

Then

$$\omega^2 V_1^{(m)}(-2q_r) - V_2^{(m)}(-2q_r) = -(2\pi)^{\frac{1}{2}} A_m(\omega,p_r|q_r) \qquad (26d)$$

and finally

$$T_m(\omega,p_r|q) = A_m(\omega,p_r|q) + \frac{1}{(2\pi)^{\frac{1}{2}}}\left[\omega^2 V_1^{(m)}(-q_r-q) - V_2^{(m)}(-q_r-q)\right] \qquad (26e)$$

A Very Simple Example

Suppose density is constant everywhere. For z < 0, $v = v_o$. For z > 0, $v = v_f$. In this case we need only look at $\rho_r = \rho_s = 1$. The data field $D = G - G_o$ has at the earth's surface

$$\langle o|D(\omega)|o\rangle = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{iv_o}{2\omega} R = \frac{iv_o}{2\omega} \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{v_f - v_o}{v_f + v_o} \qquad (27)$$

The potential V is

$$V(\omega,z) = \omega^2\left(\frac{1}{v_f^2} - \frac{1}{v_o^2}\right) \qquad z > 0$$

$$= 0 \qquad z < 0$$

with Fourier transform

$$V(\omega,q) = \omega^2 V(q) = \frac{\omega^2}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \left(\frac{1}{v_f^2} - \frac{1}{v_o^2}\right) dz\, e^{-iqz}$$

$$= \frac{-\omega^2}{(2\pi)^{\frac{1}{2}}} \frac{i}{q_-} \left(\frac{1}{v_f^2} - \frac{1}{v_o^2}\right) \qquad (28)$$

(q_- is short for $\lim_{\epsilon \to 0} q - i\epsilon$.) Let's see if this potential can be recovered from the data.

The first approximation to V is the Born approximation (26a):

$$D_o(\omega) = \frac{-v_o^2}{8\pi\omega^2} \omega^2 V^{(1)} \left(\frac{-2\omega}{v_o}\right)$$

or

$$V^{(1)}(q) = \frac{4i}{(2\pi)^{\frac{1}{2}} v_o^2 q_-} \left(\frac{v_f - v_o}{v_f + v_o}\right) = \frac{4i}{(2\pi)^{\frac{1}{2}} v_o^2 q_-} R \qquad (29a)$$

in the limit as $v_f \to v_o$, $V^{(1)} \to V$.

To attain a better approximation, we get

$$T_1(\omega|q) = \frac{\omega^2}{(2\pi)^{\frac{1}{2}}} V^{(1)} \left(\frac{-\omega}{v_o} - q\right) = \frac{-2i\omega^2}{\pi v_o \left(q + \frac{\omega}{v_o}\right)_+} R \qquad (29b)$$

In this case, the integral equation (26c) can be evaluated analytically, yielding

$$V^{(2)}(q) = \frac{-8i}{(2\pi)^{\frac{1}{2}} v_o^2 q_-} R^2 \qquad (29c)$$

$$T_2(\omega|q) = \frac{-2i}{\pi} \frac{\frac{\omega}{v_o} - q}{\left(\frac{\omega}{v_o} + q\right)^2} \frac{\omega^2}{v_o^2} R^2 \tag{29d}$$

$$V^{(3)}(q) = \frac{12i}{(2\pi)^{\frac{1}{2}}} \frac{1}{v_o^2 q_-} R^3 \tag{29e}$$

$$T_3(\omega|q) = \frac{-2i}{\pi} \frac{\left(\frac{\omega}{v_o} - q\right)^3}{\left(\frac{\omega}{v_o} + q\right)^3} \frac{\omega^2}{v_o^2} R^3 \tag{29f}$$

and so on. Our iterative method has given us in this case a power series in the reflection coefficient R. For reasonable values of R, convergence would appear to be rapid.

The Scheme 1 iterative method can also be done analytically in this case. The first approximation to V is just the Born approximation (29a). The second approximation is

$$V(q) \simeq \frac{4i}{(2\pi)^{\frac{1}{2}} v_o^2 q_-} \frac{R}{1 + 2R}$$

which is actually slightly better than the third Scheme 2 approximation.

Conclusions

We seem to have developed by brute force an inversion algorithm for seismic data. The stability of the algorithm is open to question, and the economics of it we would rather not talk about. For the one simple example given above, the analytic properties of T and V allowed a solution by contour integration. If these analytic properties were to hold in general (we have no idea if they do) then one of the integrals in (20) or (25) might be done analytically, in which case the algorithm might even be practical.