GILDING THE BORN APPROXIMATION

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In its simplest (and most computable) form the Born approximation requires the user to input a constant background velocity. Since the Born approximation propagates waves at the background velocity, it can result in a poor estimation of both the location and magnitude of velocity changes. A variable background velocity may be approximated (at least for the scalar wave problem) by stretching the time axis, but that gives rise to other problems. In any event, multiple reflections are treated as if they didn't exist.

Some of the shortcomings of the simple Born approximation can be overcome at a cost, by going to its "distorted wave" form. Basically this amounts to using as the "unperturbed" wave equation a variable-parameter equation with known solutions. For example, since it appears we may solve for a layered medium using the Gelfand-Levitan technique, suppose we have derived an estimate $\bar{K}(z), \bar{\rho}(z), \bar{v}(z) = \sqrt{\bar{K}/\bar{\rho}}$ of "average" bulk modulus, density and velocity versus depth. We may then calculate fairly rapidly an impulse response G_0 to use in the Born approximation to the real impulse response G. Here's how it might work: Start with the real wave equation

$$\left(\nabla \cdot \frac{1}{\rho} \nabla + \frac{\omega^2}{K}\right) \psi = 0 \tag{1}$$

with the unknown coefficients K and ρ. We wish to put (1) into the form of a Schroedinger equation, plus a potential term. We make a depth-to-traveltime conversion using the velocity estimate $\bar{v}(z)$:

$$\tau = \int_0^z \frac{dz}{\bar{v}(z)} \tag{2}$$

in terms of which the wave equation is

$$\left[\partial_x \frac{1}{\rho} \partial_x + \frac{1}{\bar{v}} \partial_\tau \frac{1}{\rho\bar{v}} \partial_\tau + \frac{\omega^2}{K}\right] \psi = 0$$

Now, define the parameter $\eta(x,z)$ by

$$\eta = \frac{1}{\sqrt{\rho\bar{v}}} \tag{3}$$

and rescale the wave function ψ as

$$\phi = \eta \psi$$

Then, the wave equation for ϕ is

$$\left[\bar{v}\ \partial_x\ \eta^2\ \partial_x \frac{1}{\eta} + \frac{1}{v} \partial_\tau\ \eta^2\ \partial_\tau \frac{1}{\eta} + \frac{\omega^2}{K\eta}\right] \phi = 0$$

Now we invoke the identity

$$\partial_\zeta\ \eta^2\ \partial_\zeta \frac{\phi}{\eta} = \eta\ \phi_{\zeta\zeta}\ \phi - \eta_{\zeta\zeta}\ \phi$$

where ζ may be either x or τ. The wave equation for ϕ becomes

$$\left[\bar{v}^2\ \partial_x^2 + \partial_z^2 + \omega^2 \frac{\bar{v}^2}{v^2} - \frac{1}{\eta} (\bar{v}^2\ \eta_{xx} + \eta_{\tau\tau})\right] \phi = 0 \tag{4}$$

Now the system we can supposedly solve is

$$\left(\bar{v}^2\ \partial_x^2 + \partial_\tau^2 + \omega^2 - \frac{\bar{\eta}_{\tau\tau}}{\bar{\eta}}\right) \phi_0 = 0 \tag{5}$$

with

$$\bar{\eta} = \frac{1}{\sqrt{\bar{\rho}\bar{v}}} \tag{6}$$

So, we define as our unperturbed wave operator

$$L_0(\omega) \quad v^2\, \partial_x^2 + \partial_\tau^2 - \frac{\bar{\eta}_{\tau\tau}}{\bar{\eta}} + \omega^2 \tag{7}$$

and as a potential

$$V(\omega,x,\tau) = \omega^2\left(\frac{\bar{v}^2}{v^2} - 1\right) + \frac{\bar{\eta}_{\tau\tau}}{\bar{\eta}} - \frac{\bar{v}^2\ \eta_{xx} + \eta_{\tau\tau}}{\eta} \tag{8}$$

It will prove convenient to distinguish between the frequency-dependent and frequency-independent parts of the potential:

$$V(\omega,x,\tau) = \omega^2 V_1(x,\tau) + V_2(x,\tau) \tag{9}$$

It will also prove useful to have a Fourier transform $(x \rightarrow p)$ of the potential handy:

$$\tilde{V}(\omega,p,\tau) = \frac{1}{\sqrt{2\pi}} \int dx\ e^{-ipx}\ \tilde{V}(\omega,x,\tau)$$

$$= \omega^2 \tilde{V}_1(p,\tau) + \tilde{V}_2(p,\tau) \tag{10}$$

The Born approximation requires that we construct an impulse response $G_0(\omega)$ for the unperturbed wave operator L_0. This is most easily done if we take a Fourier transform over x of L_0, since G_0 will be diagonal in the p representation. G_0 will have the form

$$\langle p_r,\tau_r | G_0(\omega) | p_s,\tau_s \rangle = \delta(p_r - p_s)\ G_0(\omega,p_r;\tau_r|\tau_s) \tag{11}$$

with G_0 a solution of

$$\left[\partial^2_{\tau_r} + \omega^2 - \frac{\bar{\eta}_{\tau_r\tau_r}}{\bar{\eta}} - \bar{v}^2 p_r^2\right] G_0\ (\omega, p_r; \tau_r | \tau_s) = -\delta(\tau_r - \tau_s) \tag{12}$$

Given G_0 and V, the Born approximation then allows us to construct an approximation to the real impulse response G, or

$$G = G_0 + G_0\, V\, G_0$$

In the seismic experiment, we measure the response at a point $(x_r, \tau_r{=}0)$ to an impulse at a point $(x_s, \tau_s{=}0)$. The direct arrival is hopefully filtered out, leaving a data field

$$D(\omega, x_r | x_s) = \langle x_r, \tau_r{=}0 | G - G_0 | x_s, \tau_s{=}0 \rangle$$

$$= \langle x_r, 0 | G_0 V G_0 | x_s, 0 \rangle \tag{13}$$

It is convenient to Fourier transform $x_r \rightarrow p_r$, $x_s \rightarrow p_s$ in this relation, giving

$$\bar{D}\ (\omega, p_r | p_s) = \langle p_r, 0 | G_0 V G_0 | p_s, 0 \rangle$$

Since G_0 is diagonal in p, and v is diagonal in τ, this equation takes a simple integral form:

$$\bar{D}\ (\omega, p_r | p_s) = \frac{1}{\sqrt{2\pi}} \int d\tau\ G_0\ (\omega, p_r; 0 | \tau)\ \bar{V}(\omega, p_r - p_s, \tau)\ G_0(\omega, p_s; \tau | 0) \tag{14}$$

which is really less complicated than it looks. We can simplify the form of (14) as follows:

Define a source-receiver midpoint wavenumber $P = p_r - p_s$

Define a source-receiver offset wavenumber $p = p_r + p_s$

Redefine the data field in terms of these new coordinates:

$$D(\omega,P,p) = \bar{D}\left[\omega, \frac{P+p}{2} \mid \frac{p-P}{2}\right]$$

Define a transformation matrix F:

$$F_2(\omega,P,p|\tau) = \frac{1}{\sqrt{2\pi}} G_0(\omega,\frac{p+P}{2};0|\tau) \cdot G_0(\omega,\frac{p-P}{2};\tau|0)$$

$$F_1(\omega,P,p|\tau) = \omega^2 F_2(\omega,P,p|\tau)$$

Put all that in (14) and we get [remembering the decomposition (9) for V]

$$D(\omega,P,p) = \sum_{\alpha=1}^{2} \int d\tau \; F_\alpha(\omega,P,p|\tau) \; \bar{V}_\alpha(P,\tau) \qquad (15)$$

For every value of P, the integral equation (15) must be invented to find $\bar{V}_1$ and $\bar{V}_2$. Counting indices we seem to have $N\omega \times Np$ equations with $2 \times N\tau$ unknowns, indicating, as for the simple Born approximation, an overdetermined problem.

There is no reason to suppose that the distorted wave approach wouldn't work, or that reasonably economical means of inverting equation (15) wouldn't be found. [It is not even necessary that we require the "unperturbed" system to be a layered medium, provided we can solve the forward problem for it, and provided we are willing to put a second integral (over P) into (15).] It is not clear, however, exactly what we have. Since G_0 in this case contains multiple reflections, so will G, and the solution of (15) for V incorporates an attempt to identify and remove them. However, all waves will be propagating with the estimated velocity $\bar{v}$, so it is hard to imagine the technique really doing a good job of identifying and removing multiples.

There are other games one can play with the Born approximation. If one is indifferent to multiples, but wants to follow the true velocity structure closely, approximate "one-way" Green's functions can be constructed using finite-difference or finite-element techniques. Using these Green's functions in the Born approximation amounts to firming up the squishy soft notions of reflectivity used in contemporary migration schemes.