WIDE-ANGLE VARIABLE-VELOCITY ONE-WAY WAVE-EQUATION MODELING

Bert Jacobs

Abstract

Phase-shift techniques for modeling the one-way wave equation can be generalized to generate synthetic seismograms due to media with laterally varying velocity. These synthetics will not have the dip dispersion associated with finite-difference algorithms.

Introduction

Good synthetic seismograms are needed in the testing of migration packages. Synthetics for this purpose are often computer-generated by a modeling algorithm whose accuracy is known to exceed that of the migration routine. Such modeling algorithms should be based on the one-way wave equation, since it is obvious that present-day migration programs do not correctly handle multiple reflections. Good behavior at high dip is also a necessity. Both these constraints are satisfied by phase-shift algorithms when we treat of media which are laterally invariant. Laterally changing media present a problem which can be solved by suitably generalizing these algorithms.

Phase-Shift Methods

In z-variable media the phase-shift algorithm takes a wave field $U(k_x,\omega,z+\Delta z)$ and a source $S(k_x,\omega,z)$ and computes the wave field at depth z by implementing a discrete form of the continuous equation

$$U(k_x,\omega,z) = U(k_x,\omega,z+\Delta z)\ \exp\left\{-i\Delta z\left[\left(\frac{\omega}{v(z)}\right)^2-k_x^2\right]^{\frac{1}{2}}\right\} + S(k_x,\omega,z) \qquad (1)$$

where v(z) is half the acoustic velocity of the earth.

This method can be generalized by turning the wave-equation modeling problem into an eigenvalue problem. The eigenvalues of the new problem will be the p_z's of the phase shifts needed in wave equation approximation.

The first step in the derivation is to notice that the one-way wave equation is easily fooled. If we are pushing the wave field between depths $z+\Delta z$ and z and the velocity is independent of z within this region then the solution to the one-way wave equation will be independent of the velocity structure outside this narrow strip of earth. This allows us to take a Fourier transform over z at every z-step.

Discretizing and taking transforms over x,z, and t we can change the wave equation

$$\partial_{xx}U(k_x,\omega,z) + \partial_{zz}U(k_x,\omega,z) = \frac{1}{v(x)^2}\,U(k_x,\omega,z)$$

into the convolutional equation

$$\left(\Delta k_x^2 k^2+p_z^2\right)\ U_k(p_z,f) = \Delta\omega^2 f^2 W_k \delta(p_z) ** U_k(p_z,f) \qquad (2)$$

where W_k is the Fourier transform of the square of the slowness along the x-axis at depth z. The transform variable p_z is continuous while k and f are integers which index discrete spatial (horizontal) and temporal wavenumbers, respectively. If we introduce the summation convention of tensor notation then equation (2) becomes

$$\left(\Delta\omega^2 f^2 W_{k-n} - \Delta k_x^2 k^2 \delta_{kn}\right)\ U_k(p_z,f) = p_z^2 U_k(p_z,f) \qquad (3)$$

Equation (3) is shorthand for a system of equations with an easy, though trivial solution, i.e. $U_k(p_z,f) = 0$. We will want a non-trivial nonzero set of solutions. The system of equations in (3) is not square so we need to ditch some of the equations (or variables). There is probably little harm done if the variables and equations corresponding to high spatial and temporal frequencies are left out of the problem, so we will assume from now on that this has been done. Under these conditions, system (3) will have a nontrivial solution when p_z^2 is an eigenvalue of the matrix of coefficients that resides there.

The matrix of our eigenvalue problem is banded and Hermitian so it has a complete set of orthonormal eigenvectors with associated real eigenvalues. Our matrix is almost certainly not positive definite since it is expected that evanescent waves will have negative eigenvalues. This won't bother us much. Taking a hint from commonly used F-K technique we will just project the evanescent energy out of the picture. Letting the integer λ subscript the eigenvalues of the matrix in equation (3), denote the eigenvectors by $\phi_\lambda(k,f)$. With this convention the input wave field $U_k(f)$ can be decomposed into a linear combination of the $\phi_\lambda(k,f)$'s. We have

$$U_k(z+\Delta z,f) = \sum_{\lambda=1}^{N\lambda} C_\lambda(f)\phi_\lambda(k,f) \tag{4}$$

$$C_\lambda(f) = \sum_{k=1}^{Nk} U_k(z+\Delta z,f)\phi_\lambda^*(k,f) \tag{5}$$

where $N\lambda$ is the number of positive eigenvalues.

If by P we denote the projection operator which rids our wave field of its evanescent energy and by G the operator of equation (3) then what we have in equations (4) and (5) is a way of decomposing wave fields in terms of the eigenvectors of GP. This operator product is positive semidefinite so it has a square root. The square root has the same eigenvectors and its eigenvalues are simply the square roots of those of GP. We can use this information to construct $U_{kf}(z)$ in terms of the $\phi_\lambda(k,f)$'s. Taking a hint from the phase-shift method we propose the decomposition

$$U_k(z,f) = \sum_{\lambda=1}^{N\lambda} C_\lambda(f)\phi_\lambda(k,f)\ \exp\left[-i\Delta z\left(p_z\right)_\lambda\right] + S_k(f) \tag{6}$$

where the $C_\lambda(f)$'s are the same as those in equation (5) and the $S_k(f)$ is a source term which models the reflection coefficients at depth z.

Conclusions

A method has been constructed which is able to correctly model much of the behavior of solutions of the exact one-way wave equation. One problem with this method is that it dumps all evanescent energy at every z-step, thus losing much of the high-angle reflection seismogram as well. It might be possible to extend the phase-shift method still further by expanding the wave field in terms of the eigenfunctions of G rather than of GP. One might guess that the eigenfunctions of $G^{1/2}$ are the same as the eigenfunctions of G and that the eigenvalues of the square-root operator, which we denote by η_λ, are given by

$$\eta_\lambda = \left(k_z^2\right)_\lambda^{\frac{1}{2}}, \qquad \text{if} \left(k_z^2\right)_\lambda \geq 0$$

$$\eta_\lambda = i\left(-k_z^2\right)_\lambda^{\frac{1}{2}}, \qquad \text{if} \left(k_z^2\right)_\lambda < 0$$

This sort of trick seems to work when the phase-shift method is applied in x-invariant media (Thorson,J., SEP-16, p. 299-309).