Chapter III

MODELS FOR FREQUENCY-DEPENDENT Q

An internally consistent model for the anelasticity and wave propagation of a linear material with Q exactly independent of frequency has been given. The constant Q model fits the properties of most rocks well, especially over the two or three orders of magnitude in frequency involved in most experiments. Considering the complexity of solids, and rocks in particular, there is, however, no reason to believe that the Q of all rocks is exactly independent of frequency, nor that there is any other simple universal law that describes it. All of the previous nearly constant Q models (except for Strick's 3- and 4-parameter models [Strick, 1967]) had sharp cutoffs on Q, that have never been observed in rocks. In this chapter I will show how the constant Q (CQ) model may be generalized to include the effects of arbitrarily weak variations of Q with frequency. This generalization should be useful in correlating data over wide ranges of frequency, such as from ultrasonic pulse measurements to the seismic band or from the seismic range to the time scales involved in the Chandler wobble, post-glacial and post-seismic rebound, and tectonics.

Theory

Figure 3.1 shows four simple viscoelastic models that are treated in most standard texts on viscoelasticity. Table 3.1 gives a summary of the properties of those models, and 1/Q is plotted in figure 3.2. We have used the definition of Q as the ratio between the real and imaginary parts of the complex modulus function [O'Connell and Budiansky, 1978]. The Voigt and the Maxwell models are specified by two parameters and the other two by three. All the models feature a 1/Q that is proportional to either frequency or inverse frequency over most of the frequency ranges shown.

The CQ model can be specified by a modulus function of the form

$$M(\omega) = M_0(i\omega)^{\beta} \tag{3.1}$$

where β is related to Q by

$$Q = \cot(\frac{\pi\beta}{2}) \tag{3.2}$$

and β is in the range

$$0 \leq \beta \leq 1 \tag{3.3}$$

The limiting cases are classical elasticity and Newtonian viscosity. It is clear that physical realizability is maintained if the viscous elements in the models shown in figure 3.1 are replaced by constant Q elements. Table 3.2 lists a summary of the resulting 3- and 4-parameter models. The expressions in table 3.2 reduce to those in table 3.1 when $\beta = 1$. Examples of the frequency-dependence for these models are shown in figures 3.2 through 3.5.

It is apparent from the figures that a wide range of data with Q smoothly varying with frequency can be fitted with these simple models. Should these be insufficient to fit a particular set of observations the models can be expanded as needed but given the experimental difficulties in measuring Q it is unlikely that this will happen often.

Property / Model	Modulus $M(\omega)$ $M(\omega)$	Q	Limit $\omega \to 0$	Limit $\omega \to \infty$
Maxwell	$M_o \frac{i\omega_1}{1 + i\omega_1}$	ω_1	Viscous	Elastic $M(\omega) = M_o$
Voigt	$M_o[1 + i\omega_1]$	$\frac{1}{\omega_1}$	Elastic $M(\omega) = M_o$	Viscous
3 parameter solid	$M_o \frac{1 + (1+\varepsilon)i\omega_1}{1 + \varepsilon + i\omega_1}$	$\frac{1+\varepsilon}{\varepsilon}(\omega_1 + \omega_1^{-1})$	Elastic $M(\omega) = \frac{M_o}{1+\varepsilon}$	Elastic $M(\omega) = (1 + \varepsilon)M_o$
3 parameter fluid	$M_e \frac{i\omega_1 - \varepsilon\omega_1^2}{\varepsilon + i\omega_1}$	$\frac{1-\varepsilon^2}{\varepsilon} \frac{1}{(\omega_1 + \omega_1^{-1})}$	Viscous	Viscous

TABLE 3.1. A summary of the properties of the viscoelastic models shown in figure 3.1. Expressions are given in terms of $\omega_1 = \omega/\omega_0$ where ω_0 is the transition frequency from viscous to elastic behavior for the Maxwell and Voigt models and from Voigt to Maxwell for the other two.

Property / Model	Modulus $M(\omega)$	Q	Limit $\omega \to 0$	Limit $\omega \to \infty$
Generalized Maxwell Model	$M_o \dfrac{(i\omega_1)^{\beta}}{1 + (i\omega_1)^{\beta}}$	$Q_0 + \omega_1^{\beta}$	CQ $Q = Q_0$	Elastic $M(\omega) = M_o$
Generalized Voigt Model	$M_o[1 + (i\omega_1)^{\beta}]$	$Q_0 + \omega_1^{-\beta}$	Elastic $M(\omega) = M_o$	CQ $Q = Q_0$
Generalized 3 parameter solid model	$M_o \dfrac{1 + (1+\varepsilon)\ (i\omega_1)^{\beta}}{1 + \varepsilon + (i\omega_1)^{\beta}}$	$\dfrac{2+\varepsilon}{\varepsilon} Q_0 + \dfrac{1+\varepsilon}{\varepsilon}(Q_0^{2} + 1)^{1/2}(\omega_1^{\beta} + \omega_1^{-\beta})$	Elastic $M(\omega) = \dfrac{M_o}{1 + \varepsilon}$	Elastic $M(\omega) = (1+\varepsilon)M_o$
Generalized 3 parameter fluid model	$M_o \dfrac{(i\omega_1)^{\beta} + \varepsilon(i\omega_1)^{2\beta}}{\varepsilon + (i\omega_1)^{\beta}}$	$\dfrac{\dfrac{(Q_0^{2} + 1)^{1/2}}{\varepsilon} + Q_0(\omega_1^{\beta} + \omega_1^{-\beta}) + \dfrac{\varepsilon(Q_0^{2} - 1)}{(Q_0^{2} + 1)^{1/2}}}{\omega_1^{\beta} + \omega_1^{-\beta} + 2\varepsilon(Q_0^{-2} + 1)^{-1/2}}$	CQ $Q = Q_0$	CQ $Q = Q_0$

$Q_0 = \cot \frac{\pi\beta}{2} \quad \omega_1 = \frac{\omega}{\omega_0}$

*for $\varepsilon << 1$ this reduces to $\dfrac{(Q_0^{2} + 1)^{1/2}}{\varepsilon(\omega_1^{\beta} + \omega_1^{-\beta})} + Q_0$

TABLE 3.2. A summary of the properties of the generalized model when the viscous element has been replaced by a constant Q element.

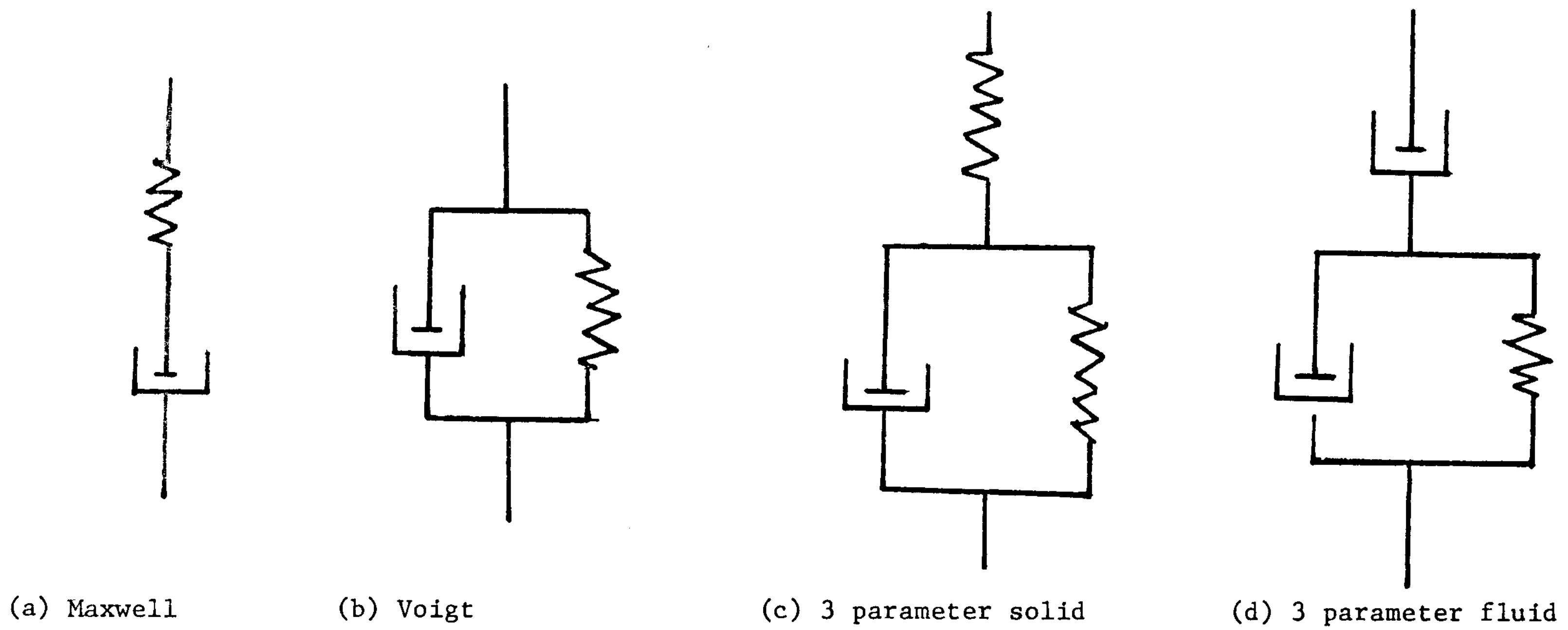

FIG. 3.1. Schematic diagrams of four simple viscoelastic models: (a) Maxwell's solid; (b) Voigt's solid; (c) 3-parameter "standard linear" solid; (d) 3-parameter fluid.

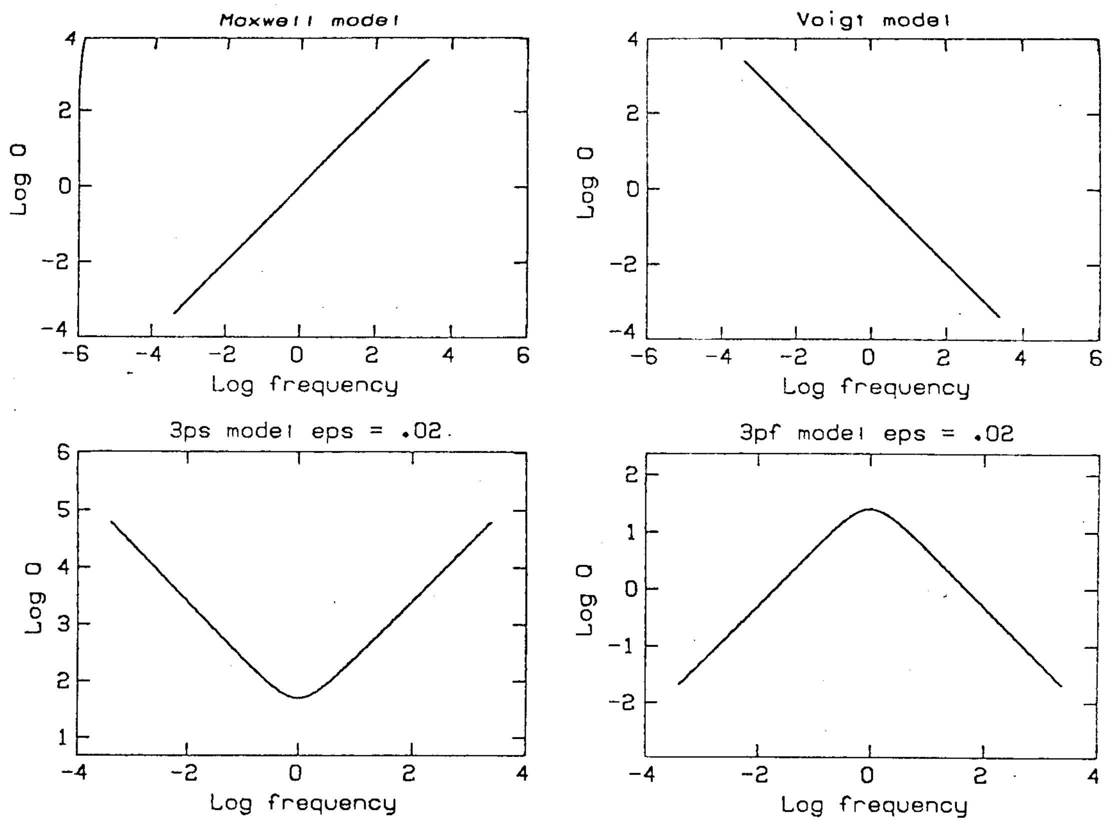

FIG. 3.2. Logarithmic plots of Q as a function of frequency relative to the transition frequency for the models shown in figure 3.1.

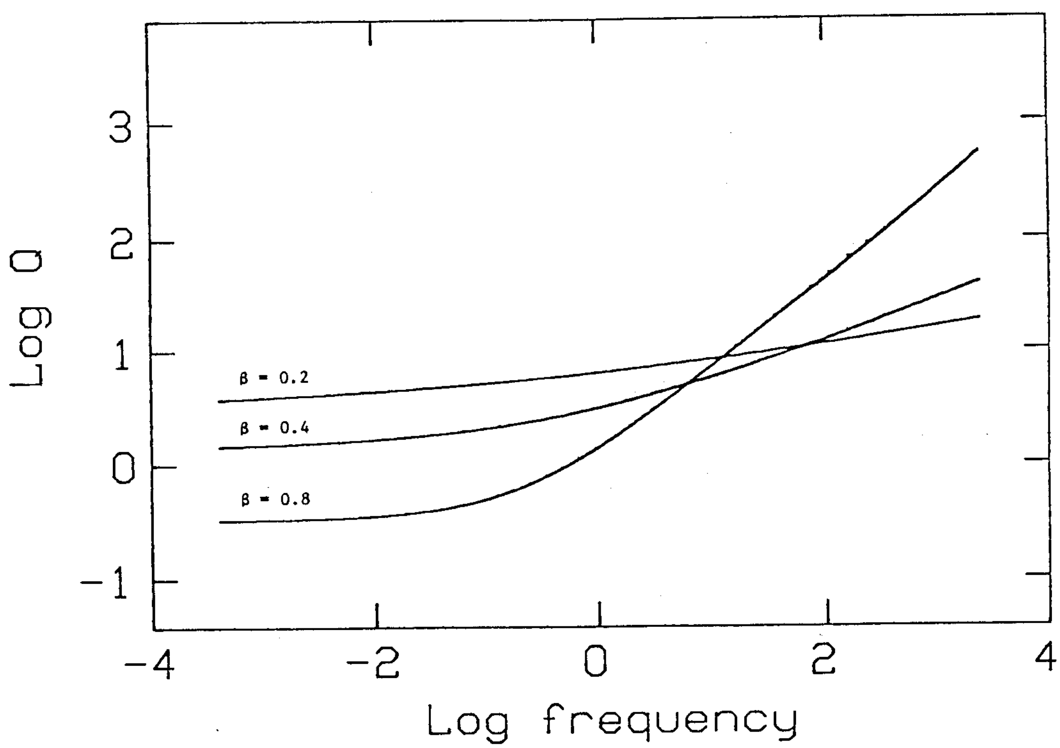

FIG. 3.3. Logarithmic plot of Q for a generalized Maxwell's model for values of β equal to 0.8, 0.4 and 0.2.

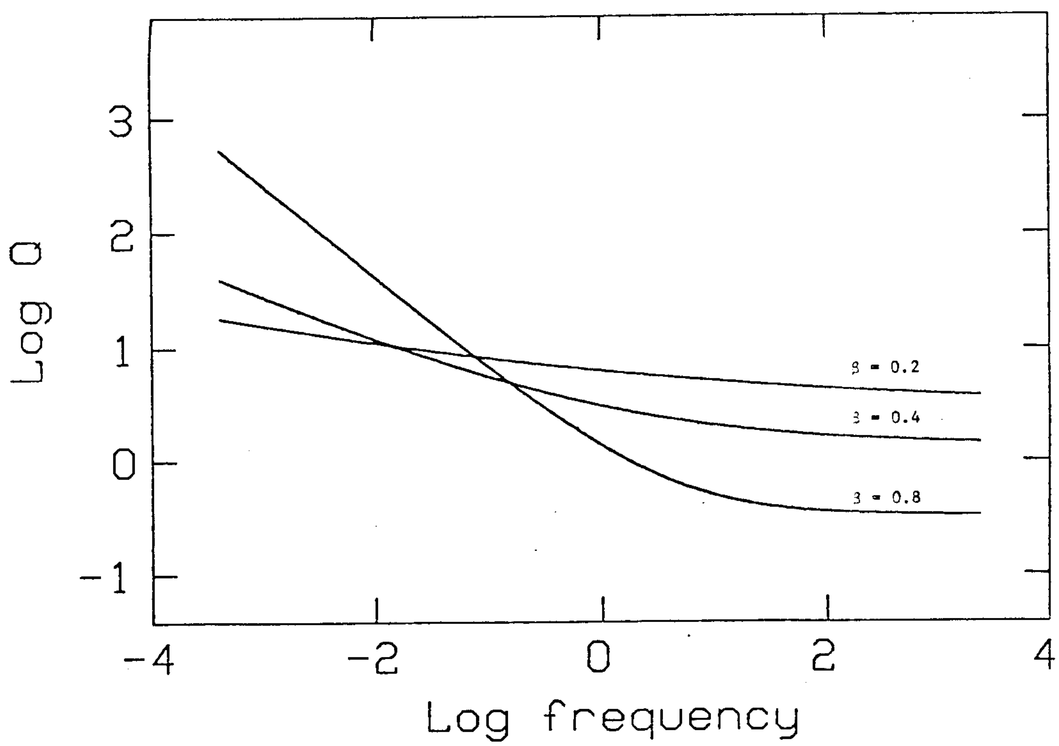

FIG. 3.4. Logarithmic plot of Q for a generalized fluid model for values of β equal to 0.8, 0.4 and 0.2.

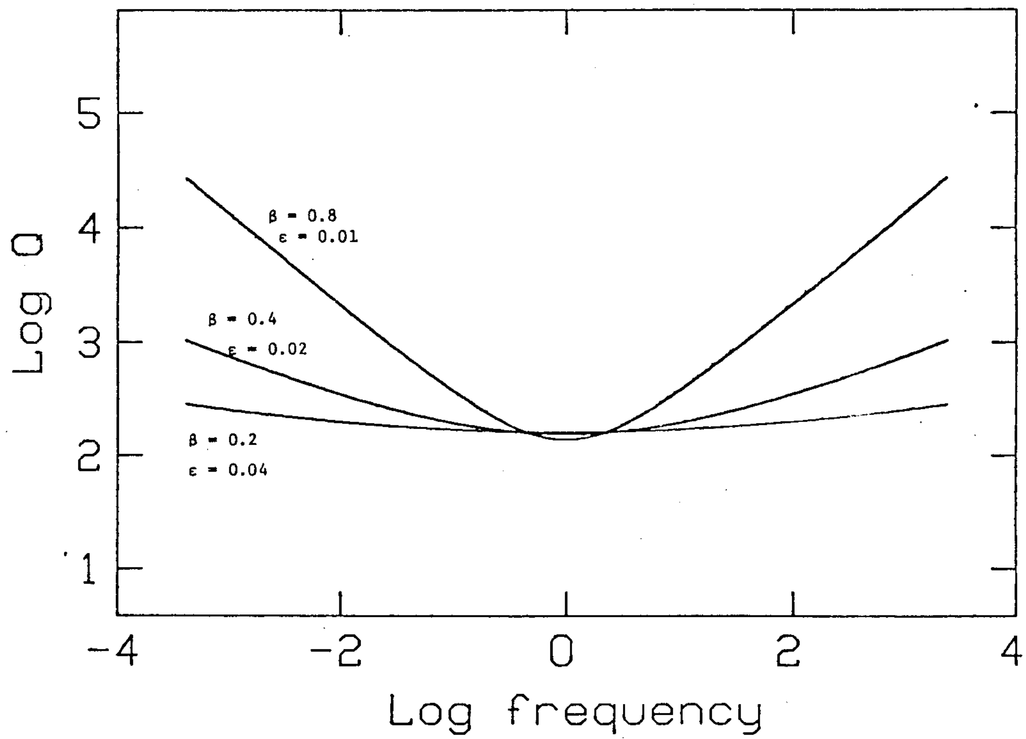

FIG. 3.5. Logarithmic plot of Q for the generalized 3-parameter solid model for β = 0.8 and ϵ = 0.01, β = 0.4 and ϵ = 0.02 and β = 0.02 and ϵ 0.04.

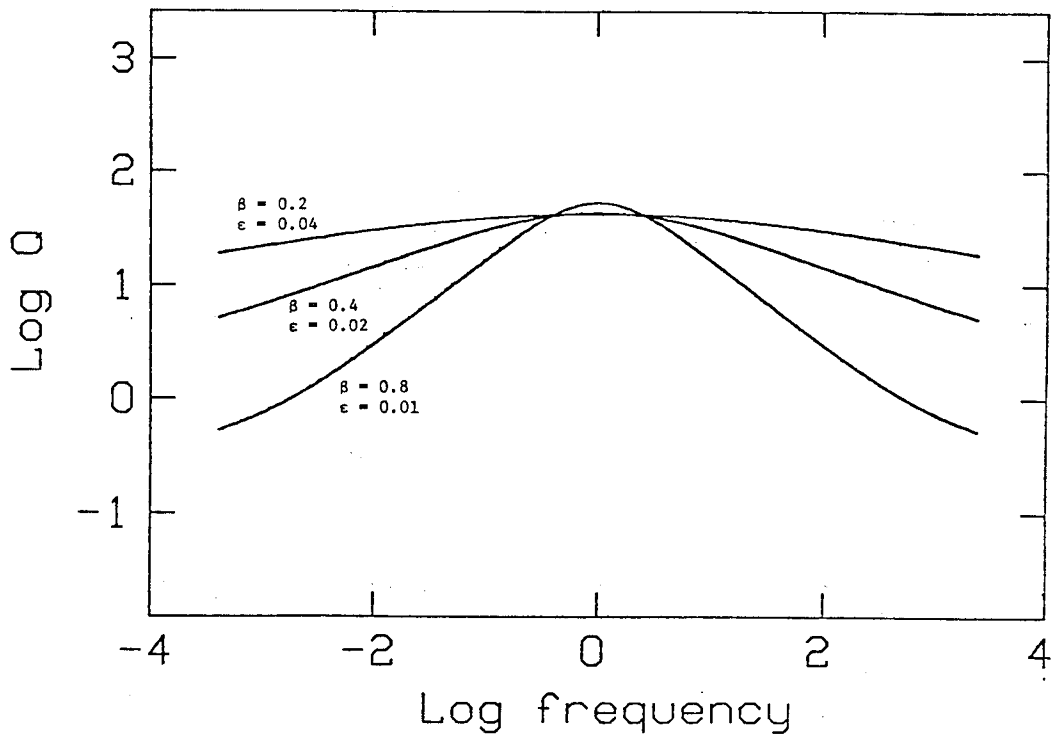

FIG. 3.6. Logarithmic plot of Q for a generalized fluid model for values of β = 0.8 and ϵ = 0.01, β = 0.4 and ϵ = 0.02, and β = 0.2 and ϵ = 0.04.