Chapter 4
3-D DOUBLE SQUARE ROOT EQUATION AND RELATED OPERATORS

It is a simple matter to extend the theory of the 2-D double square root equation to 3-D recording geometry. One can also easily develop the separable approximation (Sep) to this new DSR operator. We will see that deviation terms (Dev) present strong coupling of the two horizontal axes. Since true dips can only be properly recorded by the 3-D geometry, the Dev operator becomes more realistic in the case of 3-D than of 2-D.

4-1 3-D Development

We start with the scalar wave equation in 3-D Cartesian coordinates (x_1, x_2, z):

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right] P = 0 \tag{4-1}$$

As in the 2-D case, we assume no lateral velocity variation and transform the wavefield over (x_1, x_2, t):

$$P(k_{x_1}, k_{x_2}, z, \omega) = \iiint P(x_1, x_2, z, t)\, e^{ik_{x_1} x_1 + ik_{x_2} x_2 - i\omega t}\, dx_1\, dx_2\, dt \tag{4-2a}$$

and inversely

$$P(x_1,x_2,z,t) = \iiint P(k_{x_1},k_{x_2},z,\omega)\, e^{-(ik_{x_1}x_1 + ik_{x_2}x_2 - i\omega t)}\, dk_{x_1}\, dk_{x_2}\, d\omega \tag{4-2b}$$

Applying the differential operator (4-1) on (4-2b)

$$\frac{\partial^2}{\partial z^2} P(k_{x_1},k_{x_2},z,\omega) + \left(\frac{\omega^2}{v^2} - k_{x_1}{}^2 - k_{x_2}{}^2 \right) P(k_{x_1},k_{x_2},z,\omega) = 0 \tag{4-3}$$

For simplicity, we further assume a constant velocity medium. Then, the upcoming wave solution to (4-3) is

$$P(k_{x_1},k_{x_2},z,\omega) = P(k_{x_1},k_{x_2},0,\omega)\, e^{ik_z z} \tag{4-4}$$

where

$$k_z = -\frac{\omega}{v} \left[1 - \left(\frac{v\, k_{x_1}}{\omega} \right)^2 - \left(\frac{v\, k_{x_2}}{\omega} \right)^2 \right]^{1/2} \tag{4-5}$$

If we now imagine a 2-D array of receivers, each with a unique location (g_1, g_2) , spread over the plane (x_1, x_2) , then (4-4) can be used to downward continue these receivers. Similarly, we may consider a 2-D array of shots over the (x_1, x_2)-plane, each with a unique location (s_1, s_2) . Then, (4-4) can also be used to downward continue these shots. The total phase shift can be determined by expressing (4-5) once for receivers and once for shots, and adding the two expressions together. The 3-D counterpart of (1-15) becomes

$$DSR(G,S) = \left[1 - \left(G_1^2 + G_2^2 \right) \right]^{1/2} + \left[1 - \left(S_1^2 + S_2^2 \right) \right]^{1/2} \quad (4\text{-}6)$$

where

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \frac{v}{\omega} \begin{bmatrix} k_{g_1} \\ k_{g_2} \end{bmatrix} \quad (4\text{-}7a,b)$$

and

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \frac{v}{\omega} \begin{bmatrix} k_{s_1} \\ k_{s_2} \end{bmatrix} \quad (4\text{-}8a,b)$$

are normalized shot and receiver wavenumbers. Equation (4-6) is the double square root operator in 3-D recording geometry. Again, for simplicity, we omit the scaling wavenumber ω / v .

4-2 Midpoint-Offset Coordinates

We now make a coordinate transformation from (s_1, s_2, g_1, g_2, z, t)-space to (y_1, y_2, h_1, h_2, z, t)-space, where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} g_1 + s_1 \\ g_2 + s_2 \end{bmatrix} \tag{4-9a,b}$$

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} g_1 - s_1 \\ g_2 - s_2 \end{bmatrix} \tag{4-10a,b}$$

(y_1, y_2) are midpoint coordinates and (h_1, h_2) are (half-)offsets in inline (x_1) and crossline (x_2) directions, respectively. Using the principle of invariance of wavefields under coordinate transformation

$$P(s_1, s_2, g_1, g_2, z, t) = P'(y_1, y_2, h_1, h_2, z, t) \tag{4-11}$$

we can compute the midpoint-offset wavenumbers by applying the chain rule for partial differentiation to (4-11):

$$\frac{\partial P}{\partial s_1} = \frac{\partial P'}{\partial y_1} \frac{\partial y_1}{\partial s_1} + \frac{\partial P'}{\partial h_1} \frac{\partial h_1}{\partial s_1} \tag{4-12a}$$

$$\frac{\partial P}{\partial s_2} = \frac{\partial P'}{\partial y_2} \frac{\partial y_2}{\partial s_2} + \frac{\partial P'}{\partial h_2} \frac{\partial h_2}{\partial s_2} \tag{4-12b}$$

and

$$\frac{\partial P}{\partial g_1} = \frac{\partial P'}{\partial y_1}\frac{\partial y_1}{\partial g_1} + \frac{\partial P'}{\partial h_1}\frac{\partial h_1}{\partial g_1} \tag{4-13a}$$

$$\frac{\partial P}{\partial g_2} = \frac{\partial P'}{\partial y_2}\frac{\partial y_2}{\partial g_2} + \frac{\partial P'}{\partial h_2}\frac{\partial h_2}{\partial g_2} \tag{4-13b}$$

Using (4-9) and (4-10) we simplify (4-12) and (4-13)

$$\frac{\partial P}{\partial s_1} = \frac{1}{2}\left(\frac{\partial P'}{\partial y_1} - \frac{\partial P'}{\partial h_1}\right) \tag{4-14a}$$

$$\frac{\partial P}{\partial s_2} = \frac{1}{2}\left(\frac{\partial P'}{\partial y_2} - \frac{\partial P'}{\partial h_2}\right) \tag{4-14b}$$

and

$$\frac{\partial P}{\partial g_1} = \frac{1}{2}\left(\frac{\partial P'}{\partial y_1} + \frac{\partial P'}{\partial h_1}\right) \tag{4-15a}$$

$$\frac{\partial P}{\partial g_2} = \frac{1}{2}\left(\frac{\partial P'}{\partial y_2} + \frac{\partial P'}{\partial h_2}\right) \tag{4-15b}$$

Fourier transforming both sides of (4-14) and (4-15) and canceling

P = P' , we obtain the midpoint-offset wavenumbers in terms of shot-receiver wavenumbers :

$$\begin{bmatrix} k_{s_1} \\ k_{s_2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} k_{y_1} - k_{h_1} \\ k_{y_2} - k_{h_2} \end{bmatrix} \qquad (4\text{-}16a,b)$$

and

$$\begin{bmatrix} k_{g_1} \\ k_{g_2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} k_{y_1} + k_{h_1} \\ k_{y_2} + k_{h_2} \end{bmatrix} \qquad (4\text{-}17a,b)$$

We normalize both sides of (4-16) and (4-17)

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} Y_1 - H_1 \\ Y_2 - H_2 \end{bmatrix} \qquad (4\text{-}18a,b)$$

and

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} Y_1 + H_1 \\ Y_2 + H_2 \end{bmatrix} \qquad (4\text{-}19a,b)$$

where

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \frac{v}{2\omega} \begin{bmatrix} k_{y_1} \\ k_{y_2} \end{bmatrix} \qquad (4\text{-}20a,b)$$

and

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \frac{v}{2\omega} \begin{bmatrix} k_{h_1} \\ k_{h_2} \end{bmatrix} \qquad (4\text{-}21a,b)$$

Substituting (4-18) and (4-19) into (4-6), we obtain the 3-D double square root operator in midpoint-offset coordinates:

$$DSR = \left[1 - \left(Y_1 + H_1 \right)^2 - \left(Y_2 + H_2 \right)^2 \right]^{1/2} \qquad (4\text{-}22)$$

$$+ \left[1 - \left(Y_1 - H_1 \right)^2 - \left(Y_2 - H_2 \right)^2 \right]^{1/2}$$

4-3 The Separable Approximation

We may extend the 2-D definition of the Sep operator given by (1-39) to three-dimensional earth and obtain

$$Sep = 2 \left[1 - \left(Y_1^2 + Y_2^2 \right) \right]^{1/2} + 2 \left[1 - \left(H_1^2 + H_2^2 \right) \right]^{1/2} - 2 \qquad (4\text{-}23)$$

Equation (4-23) describes conventional processing in 3-D. The term involving H corresponds to the NMO-like operation in 3-D, and the term involving Y describes the migration of a 3-D wavefield recorded zero offset. Notice that in (4-23) Y and H are separable, which implies that, under the separable approximation, the NMO+stack and migration in 3-D can be cascaded as in 2-D. However, each operator by itself requires that all the 3-D data be handled. It is only when we make approximations that we can split each operator into two parts involving inline and crossline directions independent of each other. In fact, today's 3-D zero-offset migration utilizes the 15-degree approximation to the migration operator of (4-23), which then allows the treating of inline and crossline zero-offset sections individually.

4-4 The Deviation Operator

Again, we may utilize the formal definition of the deviation operator as the difference between the DSR and Sep operators. Using (4-22) and (4-23),

$$\mathrm{Dev} = \left[1 - \left(Y_1 + H_1 \right)^2 - \left(Y_2 + H_2 \right)^2 \right]^{1/2} \tag{4-24}$$

$$+ \left[1 - \left(Y_1 - H_1 \right)^2 - \left(Y_2 - H_2 \right)^2 \right]^{1/2}$$

$$- 2 \left[1 - \left(Y_1^2 + Y_2^2 \right) \right]^{1/2} - 2 \left[1 - \left(H_1^2 + H_2^2 \right) \right]^{1/2} + 2$$

Here, as in the case of 2-D, we would like to have estimates for H_1 and H_2 in terms of traveltime and surface offsets (h_1, h_2). We are also interested in approximating (4-24) to second order in Y as we did in Chapter 1. Using the second order square root expansion provided in Appendix A (Equation A-5), the DSR given by (4-22) takes the form

$$DSR = \left[1 - (H_1^2 + H_2^2) \right]^{1/2} \left\{ 2 - \frac{Y_1^2 + Y_2^2}{\left[1 - (H_1^2 + H_2^2) \right]} - \frac{\left(H_1Y_1 + H_2Y_2 \right)^2}{\left[1 - (H_1^2 + H_2^2) \right]^2} \right\} \tag{4-25}$$

where $1 > (H_1^2 + H_2^2)^{\frac{1}{2}} >> (Y_1^2 + Y_2^2)^{\frac{1}{2}}$.

Using the second order square root expansion given by (A-1), (4-23) takes the form

$$Sep = 2 \left[1 - \frac{Y_1^2 + Y_2^2}{2} - \frac{\left(Y_1^2 + Y_2^2 \right)^2}{8} \right]$$

$$+ 2 \left[1 - \left(H_1^2 + H_2^2 \right) \right]^{1/2} - 2$$

Ignoring terms higher than second order in Y we have

$$Sep = - (Y_1^2 + Y_2^2) + 2 \left[1 - (H_1^2 + H_2^2) \right]^{1/2} \tag{4-26}$$

If we think of $Y = (Y_1^2 + Y_2^2)^{\frac{1}{2}}$ and $H = (H_1^2 + H_2^2)^{\frac{1}{2}}$ as being the sine of offset and dip angles respectively, then (4-25) and (4-26) are 15-degree-type approximations in Y, but are of a higher degree in H. Taking the difference between (4-25) and (4-26) we obtain the approximate form of the deviation operator. Simplifying and rearranging this difference, we have the final expression

$$\text{Dev} = \left\{ 1 - \frac{1 - H_2^2}{\left[1 - (H_1^2 + H_2^2) \right]^{3/2}} \right\} Y_1^2 + \left\{ 1 - \frac{1 - H_1^2}{\left[1 - (H_1^2 + H_2^2) \right]^{3/2}} \right\} Y_2^2$$

$$- \left\{ \frac{2 H_1 H_2}{\left[1 - (H_1^2 + H_2^2) \right]^{3/2}} \right\} Y_1 Y_2 \qquad (4\text{-}27)$$

It turns out that all three terms in (4-27) are of the same order of magnitude. Perhaps, this will become more obvious when we make a second order expansion in H. Our final expression is

$$\text{Dev} = - \frac{1}{2} \left(3 H_1^2 + H_2^2 \right) Y_1^2 - \frac{1}{2} \left(H_1^2 + 3 H_2^2 \right) Y_2^2 - 2 H_1 H_2 Y_1 Y_2 \qquad (4\text{-}28)$$

Each term contains $H^2 Y^2$ — like products. Notice the strong coupling between the wavenumbers, even in the crude approximation (4-28).

Today's 3-D processing simply assumes $H_2 = 0$. For this special case (4-27) becomes

$$\text{Dev} = \left[1 - \left(1 - H_1^2 \right)^{-3/2} \right] Y_1^2 + \left[1 - \left(1 - H_1^2 \right)^{-1/2} \right] Y_2^2 \qquad (4\text{-}29)$$

and (4-28) becomes

$$\text{Dev} = - \frac{3}{2} H_1^2 Y_1^2 - \frac{1}{2} H_1^2 Y_2^2 \qquad (4\text{-}30)$$

Equations (4-29) and (4-30) imply that, under the assumption of 2-D NMO correction ($H_2 = 0$) , the 3-D deviation operator becomes separable in terms of inline and crossline terms. However, in case of true 3-D data processing ($H_2 \neq 0$), we are bound to deal with deviation operators given by (4-27) and (4-28) which are not decoupled in terms of inline and crossline terms. Perhaps the only way out is treat one of the dip operators (Y_1,Y_2) as scalar.

4-5 The Crooked Line Recording Geometry

What we would really like to derive are operators which can be implemented such that inline and crossline data can be treated independently. There may be several ways of specializing the full 3-D double square root operator (4-22) to simpler forms. Here is an example.

We consider a shooting pattern with shot points following a zig-zag pattern in the direction of the receiver axis. We treat G_1 and S_1 as operators. We also assume $H_2 = 0$. Further, we might make Y_2 a scalar function of y_2 . For instance, Y_2 = constant implies uniform cross-dip. With this assumption, (4-22) takes the form

$$DSR = \left[1 - \left(Y_1 + H_1 \right)^2 - Y_2^2 \right]^{1/2} + \left[1 - \left(Y_1 - H_1 \right)^2 - Y_2^2 \right]^{1/2}$$

Factoring the scalar $(1 - Y_2^2)^{\frac{1}{2}}$,

$$DSR = (1 - Y_2^2)^{1/2} \left\{ \left[1 - \frac{\left(Y_1 + H_1 \right)^2}{1 - Y_2^2} \right]^{1/2} + \left[1 - \frac{\left(Y_1 - H_1 \right)^2}{1 - Y_2^2} \right]^{1/2} \right\}$$

(4-31)

Notice that $(1 - Y_2^2)^{1/2}$ = cos (cross-dip) . Equation (4-31) implies that, for the crooked line geometry, we may consider the 2-D double square root operator corrected for cross-dip if we treat Y_2 as a scalar and set $H_2 = 0$.

The Sep and Dev operators for the crooked line geometry are derived below. Setting $H_2 = 0$ in (4-23) we get

$$\text{Sep} = 2\left[(1 - Y_2^2) - Y_1^2 \right]^{1/2} + 2\left(1 - H_1^2 \right)^{1/2} - 2$$

Factoring the scalar $(1 - Y_2^2)^{1/2}$

$$\text{Sep} = 2\,(1 - Y_2^2)^{1/2} \left[1 - \frac{Y_1^2}{1 - Y_2^2} \right]^{1/2} + 2\left(1 - H_1^2 \right)^{1/2} - 2 \quad (4\text{-}32)$$

Now let us derive the deviation operator. Using the second order square root expansion developed in Appendix A (Equation A-4), (4-31) takes the form

$$\text{DSR} = (1 - Y_2^2)^{1/2} \left\{ \frac{1}{(1 - Y_2^2)^{1/2}} \left(1 - Y_2^2 - H_1^2 \right)^{1/2} \right\}$$

$$\cdot \left\{ 2 - \frac{(1 - Y_2^2)\, Y_1^2}{\left(1 - Y_2^2 - H_1^2 \right)^2} \right\}$$

Simplifying,

$$DSR = 2 (1 - Y_2^2)^{1/2} \left[1 - \frac{H_1^2}{1 - Y_2^2} \right]^{1/2} \qquad (4\text{-}33)$$

$$- (1 - Y_2^2)^{-1/2} \left[1 - \frac{H_1^2}{1 - Y_2^2} \right]^{-3/2} Y_1^2$$

We expand (4-32) up to second order in Y_1

$$Sep = 2 (1 - Y_2^2)^{1/2} \left[1 - \frac{Y_1^2}{2 (1 - Y_2^2)} \right] + 2 \left[1 - H_1^2 \right]^{1/2} - 2$$

Simplifying,

$$Sep = 2 (1 - Y_2^2)^{1/2} - \left[1 - Y_2^2 \right]^{-1/2} Y_1^2 + 2 \left[1 - H_1^2 \right]^{1/2} - 2 \qquad (4\text{-}34)$$

Taking the difference between (4-33) and (4-34) and canceling the scalar-like terms, we have the final expression

$$Dev = \left[1 - \left[1 - \frac{H_1^2}{1 - Y_2^2} \right]^{-3/2} \right] \frac{Y_1^2}{(1 - Y_2^2)^{1/2}} \qquad (4\text{-}35)$$

(4-32) and (4-35), although derived for a particular 3-D geometry (the crooked line), essentially describe 2-D Sep and Dev operators corrected for cross-dip. This becomes obvious when (4-32) and (4-35) are compared with (1-40) and (2-5), respectively.

As a final note, it is important to emphasize the fact that 3-D theory is wide open to research. Here, we developed the basic theory for the 3-D double square root equation. We also introduced a challenging problem: 3-D recording geometry that can be most easily handled with equations that are yet to spring from the 3-D DSR equation.