DISPERSON RELATION DERIVATION OF WAVE EXTRAPOLATORS

A review

Jon F. Claerbout

A wave extrapolation equation is an expression for the derivative of a wavefield (usually in the depth z direction). With the wavefield and its derivative known, extrapolation can proceed by various numerical representations of $P(z + \Delta z) = P(z) + \Delta z\, dP/dz$. Two methods for finding wave extrapolation equations are the *transformation* method and the *dispersion relation* method. In the transformation method a coordinate frame is found for the scalar wave equation in which the second depth derivative $\partial_{z'z'}$ may be neglected. Then the transformed equation is solved for the first derivative term $\partial_{z'}$, giving the desired extrapolation form. In the dispersion relation method one seeks various approximations to a square root dispersion relation. Then the approximate dispersion relation is inverse transformed into a differential equation. Thanks largely to Francis Muir, the dispersion approach has evolved considerably since the writing of *Fundamentals of Geophysical Data Processing*, and it is the subject of our present review.

Substitution of the plane wave $\exp(-i\omega t + ik_x x + ik_z z)$ into the two-dimensional scalar wave equation yields the dispersion relation

$$k_z^2 + k_x^2 = \frac{\omega^2}{v^2} \tag{1}$$

Solving for k_z we get a square root

$$k_z = \frac{\omega}{v}\left(1 - \frac{v^2 k_x^2}{\omega^2}\right)^{1/2} \tag{2}$$

To inverse transform the z-axis we only need to recognize that $i k_z = \partial_z$, which means that we have an expression for a wavefield extrapolator. Regrettably, inverse transforming over x by $i k_x = \partial_x$ becomes practical only when the square root is regarded as some kind of truncated series expansion. The most popular expansion is a ratio of polynomials. Let X and R be defined by writing (2) as

$$k_z \;=\; \frac{\omega}{v}\,(1 - X^2)^{1/2} \;=\; \frac{\omega}{v}\,R \tag{3}$$

The desired polynomial ratio of order n will be denoted R_n and it will be determined by the recurrence

$$R_{n+1} \;=\; 1 - \frac{X^2}{1 + R_n} \tag{4}$$

To see what this sequence converges to (if it converges) we set $n = \infty$ in (4) and solve

$$R_\infty \;=\; 1 - \frac{X^2}{1 + R_\infty}$$

$$R_\infty(1 + R_\infty) \;=\; 1 + R_\infty - X^2$$

$$R^2 \;=\; 1 - X^2 \tag{5}$$

The square root of (5) gives the required expression (3). Actually it is only the low order terms in the expansion which are ever used.

Beginning with $R_0 = 1$ we obtain

$$5^\circ:\quad R_0 = 1 \tag{6a}$$

$$15^\circ:\quad R_1 \;=\; 1 - \frac{X^2}{2} \tag{6b}$$

$$45^\circ:\quad R_2 \;=\; 1 - \frac{X^2}{2 - \frac{X^2}{2}} \tag{6c}$$

For various historical reasons, Equations (6a,b,c) are commonly referred to as the 5-degree, 15-degree, and 45-degree equations respectively, the names giving a reasonable qualitative (but poor quantitative) guide to the range of angles that are adequately handled. A trade-off between complexity and accuracy frequently dictates choice of the approximation (6c). It then turns out that a slightly wider range of angles can be accommodated by (6c) if the recurrence is begun with something like $R_0 = \cos 45^\circ$. Accuracy enthusiasts might even have R_0 a function of velocity, space coordinates, or frequency.

Performing the substitutions of (6) into (3) we get dispersion relationships for comparison to (2):

$$5^\circ: \quad k_z = \frac{\omega}{v} \tag{7a}$$

$$15^\circ: \quad k_z = \frac{\omega}{v} - \frac{v k_x^2}{2\omega} \tag{7b}$$

$$45^\circ: \quad k_z = \frac{\omega}{v} - \frac{k_x^2}{2\frac{\omega}{v} - \frac{v k_x^2}{2\omega}} \tag{7c}$$

Graphically, these approach the semicircle of (2). They are displayed in Figure 1.

Depth-variable velocity

Identification of $i k_z$ with ∂_z converts the dispersion relations (7) into the differential equations

$$5^\circ: \quad P_z = \frac{i\omega}{v} P \tag{8a}$$

$$15^\circ: \quad P_z = i\left(\frac{\omega}{v} - \frac{v k_x^2}{2\omega}\right) P \tag{8b}$$

$$45^\circ: \quad P_z = i\left(\frac{\omega}{v} - \frac{k_x^2}{2\frac{\omega}{v} - \frac{v k_x^2}{2\omega}}\right) P \tag{8c}$$

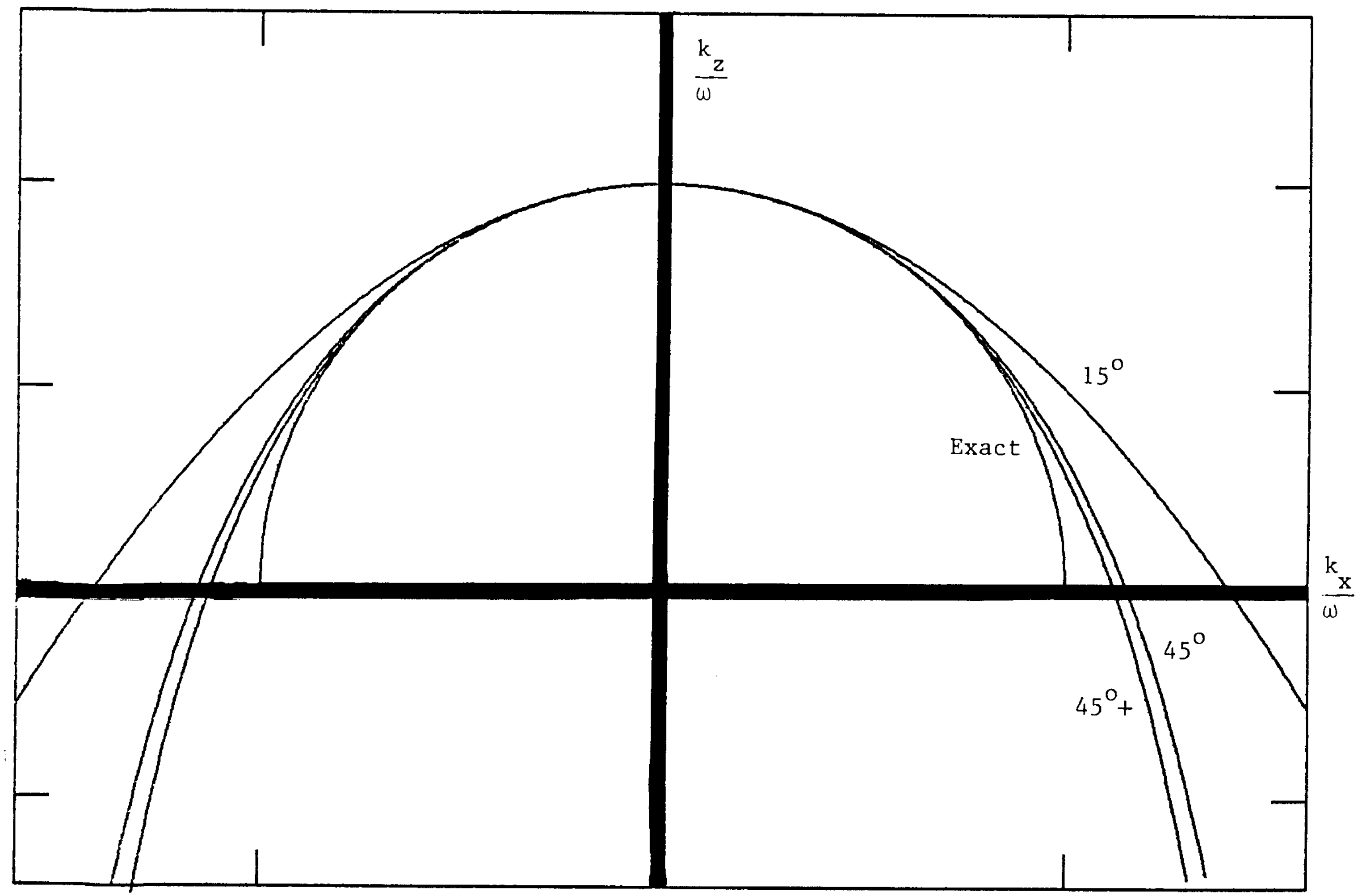

FIGURE 1.--Dispersion relation of Equations (2) and (7). The curve labeled 45°+ was constructed with $R_0 = 45^\circ$. It fits exactly at 0° and 45°.

5/80 "Original in 'Lecture-2.1'"

The differential equations (8a,b,c) were based on a dispersion relation that in turn was based on an assumption of constant velocity. So you might not anticipate that (8) has substantial validity and even great utility when the velocity is depth-variable $v = v(z)$. The actual limitations of (8a,b,c) may be characterized by their inability by themselves to describe reflection.

Retardation

Retardation is a reorganization of the wave equation so that a particular wave is handled theoretically, hence with no computational artifacts. Computational errors proportional to the grossness of the computational mesh increase with increasing departure from this particular wave, usually a normally incident plane wave. Retardation may be done in either time domain or frequency domain. In the frequency domain, we define a new variable Q where

$$P \quad = \quad Q(z) \; \exp\left[i\omega \int_0^z \frac{dz}{\bar{v}(z)} \right] \tag{9a}$$

Differentiating with respect to z we get

$$P_z \quad = \quad \exp\left[i\omega \int_0^z \frac{dz}{\bar{v}(z)} \right] \left[\partial_z + \frac{i\omega}{\bar{v}} \right] Q \tag{9b}$$

In the time domain we begin with the coordinate transformation

$$t' \quad = \quad t - \int_0^z \frac{dz}{\bar{v}(z)} \tag{10a}$$

$$z' \quad = \quad z \tag{10b}$$

$$P(z,t) \quad = \quad P'(z',t') \tag{11a}$$

The chain rule for differentiation of (11a) gives

$$P_z \quad = \quad P'_{z'} z'_z + P'_{z'} t'_z$$

$$P_z = \left[\partial_{z'} - \frac{1}{\bar{v}(z)} \quad \partial_{t'} \right] P' \tag{11b}$$

It turns out that (11a,b) are the time domain analogues of (9a,b). Note particularly that upon identification of ∂_t with $-i\omega$ the square bracket expression in (9b) equals that in (11b). Next we substitute (9) into (8) to obtain the retarded equations.

$$5^\circ: \quad Q_z = \text{zero} \qquad + i\omega \left[\frac{1}{v} - \frac{1}{\bar{v}(z)} \right] Q \tag{12a}$$

$$15^\circ: \quad Q_z = -i \frac{v k_x^2}{2\omega} Q \qquad + i\omega \left[\frac{1}{v} - \frac{1}{\bar{v}(z)} \right] Q \tag{12b}$$

$$45^\circ: \quad Q_z = -i \frac{k_x^2}{2 \frac{\omega}{v} - \frac{v k_x^2}{2}} Q \qquad + i\omega \left[\frac{1}{v} - \frac{1}{\bar{v}(z)} \right] Q \tag{12c}$$

$$\text{general:} \quad Q_z = \text{diffraction} \qquad + \text{thin lens} \tag{12d}$$

Now let us inverse transform from the horizontal wavenumber domain k_x to the horizontal space domain x by substituting $(ik_x)^2 = \partial_{xx}$. As before, the result has a wide range of validity for $v = v(x,z)$ even though the derivation would not seem to permit it. Ordinarily $\bar{v}(z)$ will be chosen to be some kind of horizontal average of $\bar{v}(x,z)$. Permitting $\bar{v}$ to become a function of x turns out to be hazardous and is rarely done.

Splitting the 45° equation

The customary numerical solution to (12c) in the x-domain is done by splitting. That is, you march forward a small Δz step alternately with the two extrapolators

$$Q_z = i\omega \left[\frac{1}{v(x,z)} - \frac{1}{\bar{v}(z)} \right] Q \tag{13a}$$

$$\left(2\,\frac{\omega}{v} + \frac{v\partial_{xx}}{2\omega}\right)\partial_z Q \;=\; +i\partial_{xx} Q \tag{13b}$$

The first of these (called the thin lens equation) may be solved analytically, whereas the second (called the diffraction equation) is solved by the Crank-Nicolson method.

Time domain

To put the above equations in the time domain, it is necessary only to get ω into the numerator and then replace $-i\omega$ by ∂_t. For example, the 15-degree, retarded, $v = \bar{v}$ equation from (12b) becomes

$$-i\omega\, Q_z \;=\; \frac{v}{2}\,(ik_x)^2\, Q$$

$$\frac{\partial^2}{\partial z\,\partial t}\,Q \;=\; \frac{v}{2}\,\frac{\partial^2}{\partial x^2}\,Q \tag{14}$$

Upcoming waves

All the above equations are for *downgoing* waves. To get equations for *upcoming* waves you need only to change the sign of z and ∂_z. Letting D denote a Downgoing wavefield and U denote an Upcoming wavefield, Equation (14), for example, takes the form

$$D_{zt} \;=\; \frac{v}{2}\,D_{xx} \tag{15a}$$

$$U_{zt} \;=\; -\,\frac{v}{2}\,U_{xx} \tag{15b}$$

It is the upcoming wave equation that almost always appears in migration problems.