EMPIRICAL TEST OF MUIR'S FRACTIONAL
DERIVATIVE REPRESENTATION THEOREM

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Francis Muir has pointed out a rather surprising result connecting the bilinear transform with Levinson recursion. He has made the (apparently correct) conjecture that:

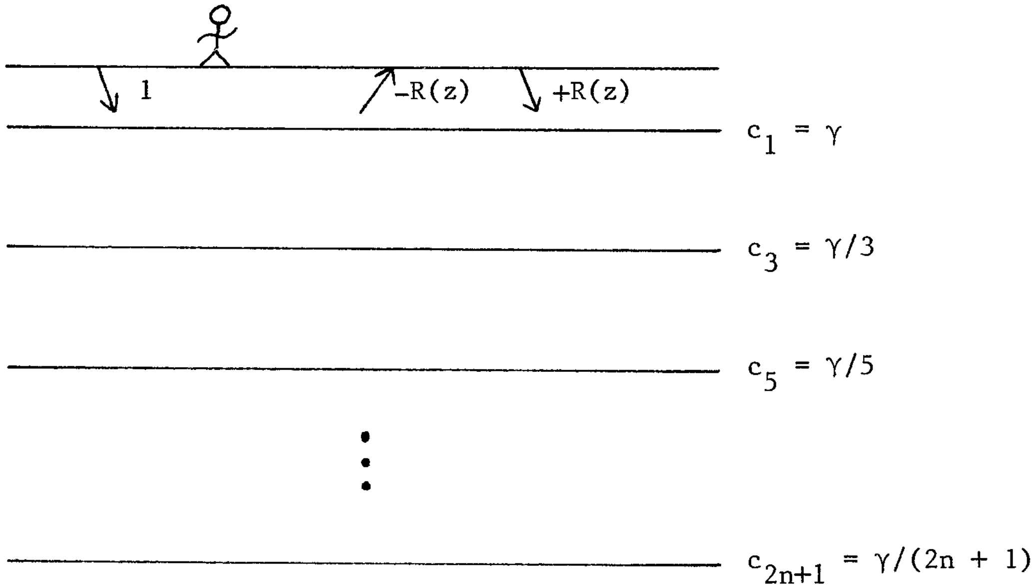
$$\left(\frac{1 - z}{1 + z}\right)^{\gamma} = \lim_{n\to\infty} \frac{A_n^-(z;\gamma)}{A_n^+(z;\gamma)} \; ; \quad -1 \le \gamma \le 1 \tag{1}$$

where $A_0^{\pm}(z;\gamma) = 1$

$$A_n^{\pm}(z;\gamma) = A_{n-1}^{\pm}(z;\gamma) \pm c_n z^n A_{n-1}^{\pm}(z^{-1})$$

$$\text{and} \quad c_n = \begin{cases} \gamma/n; & n \text{ odd} \\ 0; & n \text{ even} \end{cases}$$

A physical interpretation of this result is that the layered earth of Figure 1. has a vertical plane wave response of $[(1-z)/(1+z)]^{\gamma}$. Any model with a finite number of these layers will be a minimum phase approximation to a constant Q earth.



$$R(z) = \left(\frac{1 - z}{1 + z}\right)^{\gamma}$$ is the reflection response of this earth.

FIGURE 1.

The background material for the representation assumed in Equation (1) is covered in Chapter 8 of *Fundamentals of Geophysical Data Processing*. An argument using propagator matrices (FGDP, p. 156) shows that the impedance, R_n, of a stack of n layers is given by:

$$R_n = \frac{z^n G_n\left(\frac{1}{z}\right)}{F_n(z) - z^n G_n\left(\frac{1}{z}\right)} \tag{2}$$

where $F_n(z) = F_{n-1}(z) + c_n z \, G_{n-1}(z)$

$$G_n(z) = c_n \, F_{n-1}(z) + z \, G_{n-1}(z)$$

$$F_o = 1, \quad G_o = 0$$

If we now define $A_n(z) = F_n(z) - z^n \, G_n(1/z)$, then the Kunetz equation (FGDP, p. 158) states that

$$1 + R_n(z) + R_n\left(\frac{1}{z}\right) \quad \alpha \quad \frac{1}{A_n(z)\, A_n\left(\frac{1}{z}\right)} \tag{3}$$

One side of this autocorrelation is

$$\begin{aligned} \frac{1}{2} + R_n(z) &= \frac{\frac{1}{2}\left[F_n - z^k G_n\left(\frac{1}{z}\right)\right] + z^k G_n\left(\frac{1}{z}\right)}{F_n - z^k G_n\left(\frac{1}{z}\right)} \\ &= \frac{\left[F_n + z^k G_n\left(\frac{1}{z}\right)\right]}{2\left[F_n - z^k G_n\left(\frac{1}{z}\right)\right]} = \frac{1}{2}\,\frac{A_n^+}{A_n^-} \end{aligned} \tag{4}$$

where A_n^+ and A_n^- both satisfy the recursion indicated in (1). The reader should also note that since $|F| > |G|$ for all n, A_n^+ and A_n^- are guaranteed to be minimum phase. It may be concluded that any impedance function will always have a representation of the form of Equation (1). Muir's contribution was to correctly guess the reflection coefficients for this particular impedance function.

The representation of Equation (1) can be verified to second order in z^2 through the following analysis. Expanding the left-hand side of (1) gives

$$\begin{aligned} \left(\frac{1 - z}{1 + z}\right)^{\gamma} &= (1 - z)^{\gamma}(1 + z)^{-\gamma} = \left[1 - \gamma z + \frac{\gamma(\gamma - 1)}{2!}\, z^2 - \ldots\right] \\ &\qquad \left[1 - \gamma z + \frac{(-\gamma)(-\gamma - 1)}{2!}\, z^2 + \ldots\right] \\ &= 1 - 2\gamma z + 2\gamma^2 z^2 - \ldots \end{aligned} \tag{5}$$

If the first few steps of the Levinson recursion are carried out analytically for the expression on the right hand side of (1), a pattern emerges for the coefficients of $A_n^{\pm}$ up to order z^2. In the time domain this pattern is

$$A_\infty^{\pm} = \begin{bmatrix} 1 \\ \pm\gamma \\ \gamma^2 \left[\sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k-1)} \right] \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ \pm\gamma \\ \frac{\gamma^2}{2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad (6)$$

Now

$$\frac{A_\infty^- (z;\gamma)}{A_\infty^+ (z;\gamma)} = A_\infty^- (z;\gamma)\ A_\infty^+ (z;-\gamma)$$

$$= \left(1 - \gamma z + \frac{\gamma^2 z^2}{2} - \ldots\right) \left(1 - \gamma z + \frac{\gamma^2 z^2}{2} - \ldots\right)$$

$$= 1 - 2\gamma z + 2\gamma^2 z^2 \qquad (7)$$

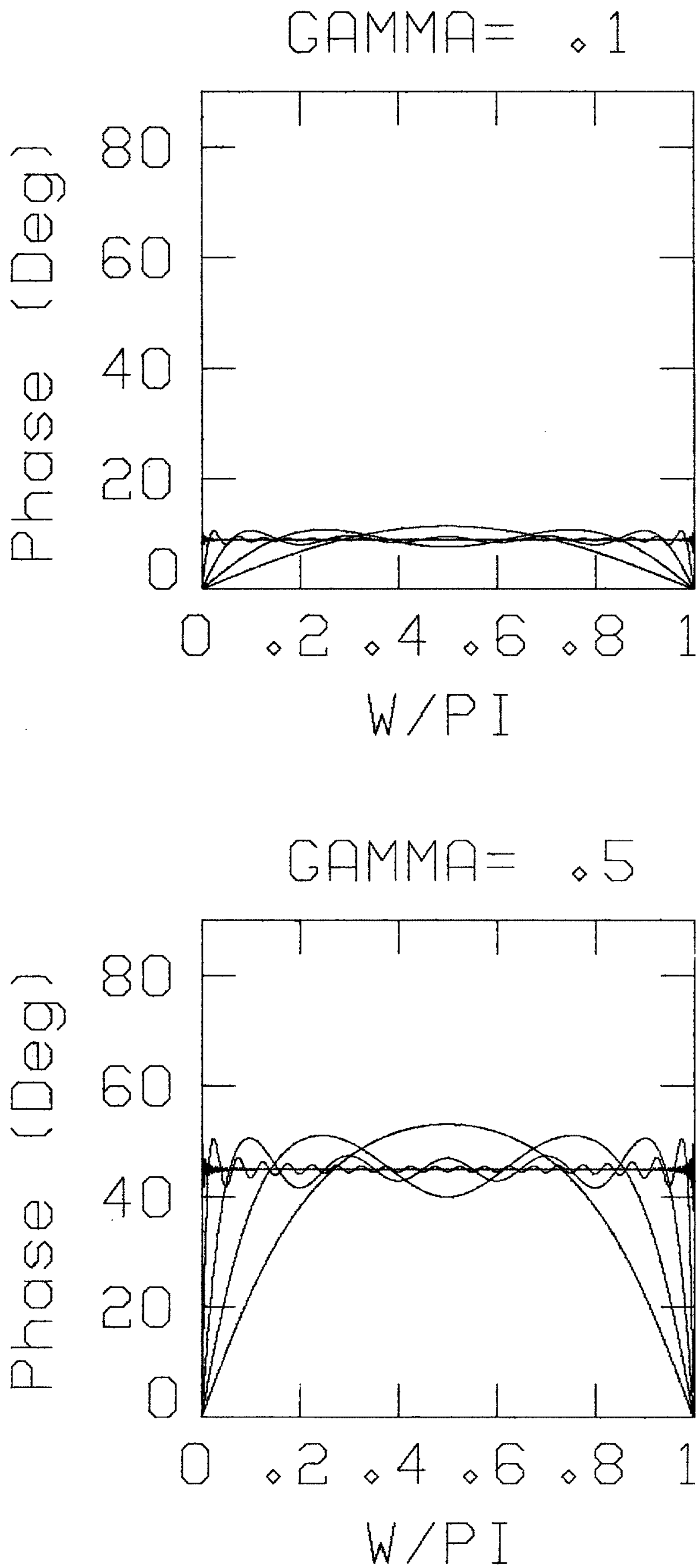
which agrees to second order in z^2 with (5).

A proof of equality to any general order in z has so far eluded this author.

To get a feel for the convergence properties of the algorithm, two suites of curves were computed. In the first set (Figures 1A and 1B) the phase of A_n^-/A_n^+ is plotted as a function of ω between 0 and π for four different values of γ between 0 and 1. The five different curves on each graph are the results of 2, 4, 10, 40 and 400 iterations. The second set (Figures 2A and 2B) is the same as the first set except that γ is now the independent variable and ω/π is fixed.

In general, convergence is fast for γ near 0 and 1 (for $\gamma = 0$ or 1 convergence is immediate for all ω) and bad for γ near 0.5. This is not surprising when we look at the binomial expansion for $(1 - z)^\gamma$ [Equation (2)]. In ω-space convergence is good for ω near $\pi/2$ and bad for ω close to 0 or π.

One interesting practical application of this is that a reasonably good time domain Hilbert transformer can be constructed by cascading approximations to $[(1 - z)/(1 + z)]^{1/2}$ and $[(1 + z^{-1})/(1 - z^{-1})]^{1/2}$. From the $\gamma = .5$ curve it can be seen that we only require ten recursions to ensure that the phase error is less than 10^{o} between 5 percent and 95 percent of the Nyquist frequency. To ensure stability the last filter has to be implemented on time reversed input in a separate stage from the first filter.


GAMMA= .1
Phase (Deg)
80
60
40
20
0
0 .2 .4 .6 .8 1
W/PI
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Phase (Deg)
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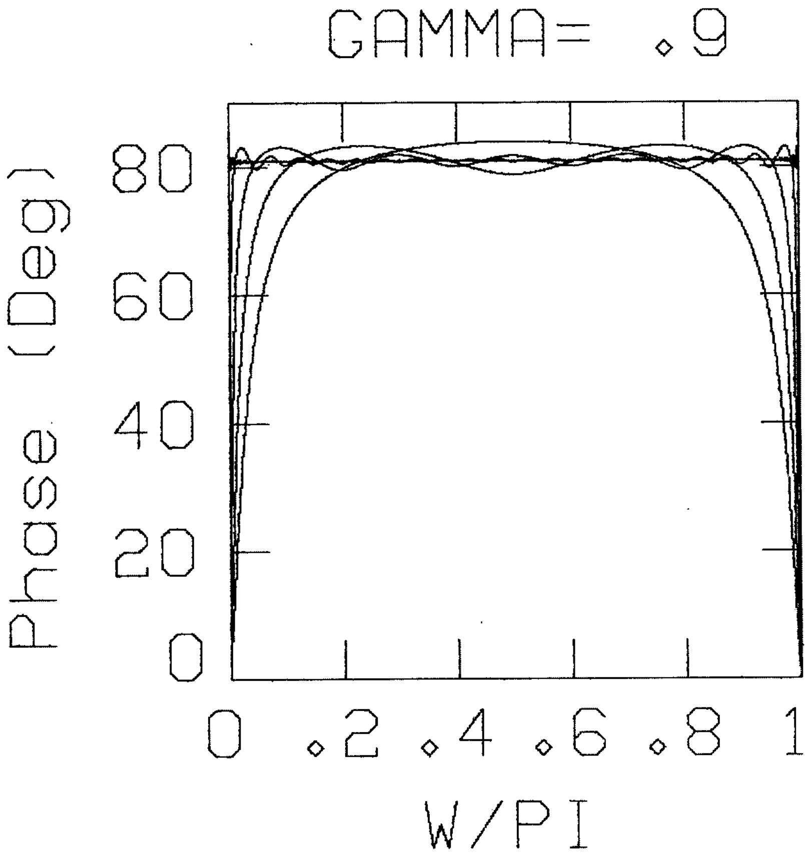
60
40
20
0
0 .2 .4 .6 .8 1
W/PI

FIGURE 1A.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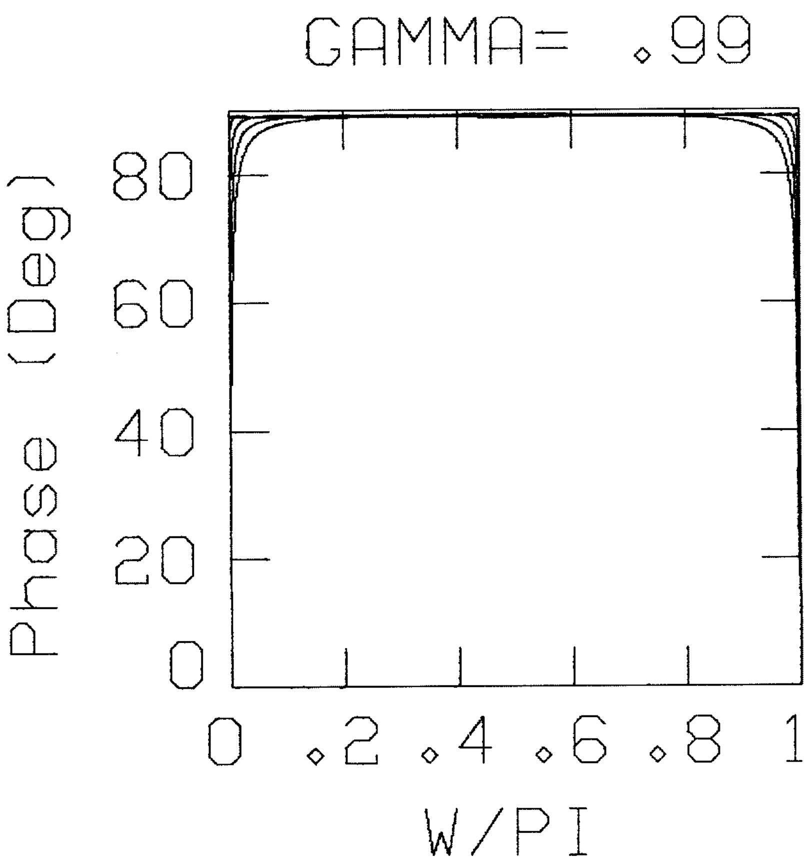
0
.2
.4
.6
.8
1
W/PI


GAMMA= .99
Phase (Deg)
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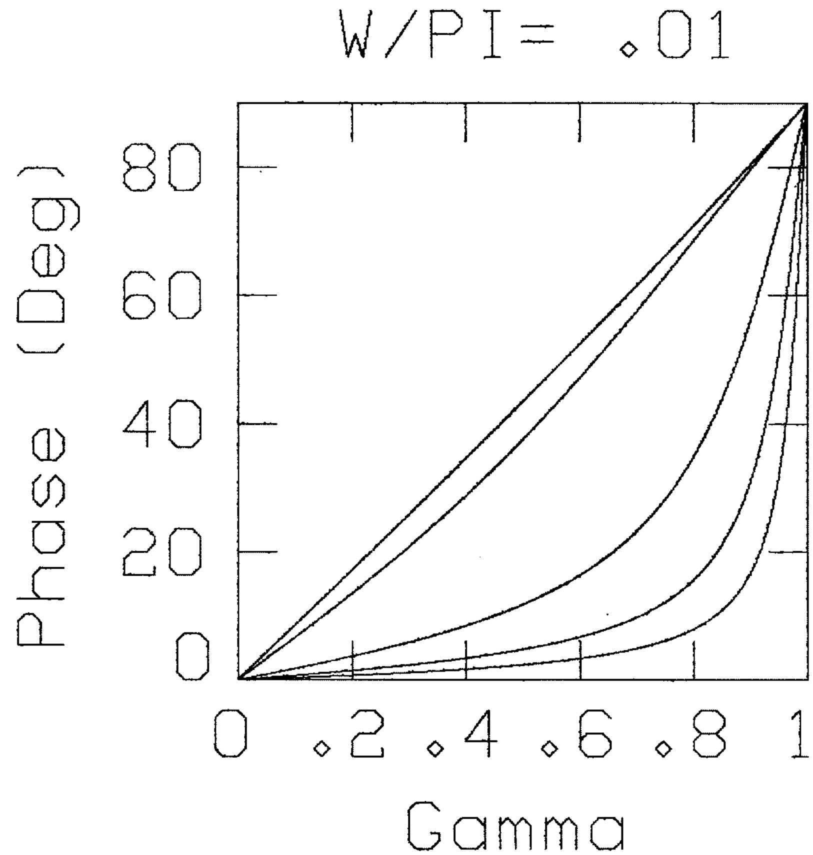
20
40
60
80
0
.2
.4
.6
.8
1
W/PI

FIGURE 1B.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Phase (Deg)
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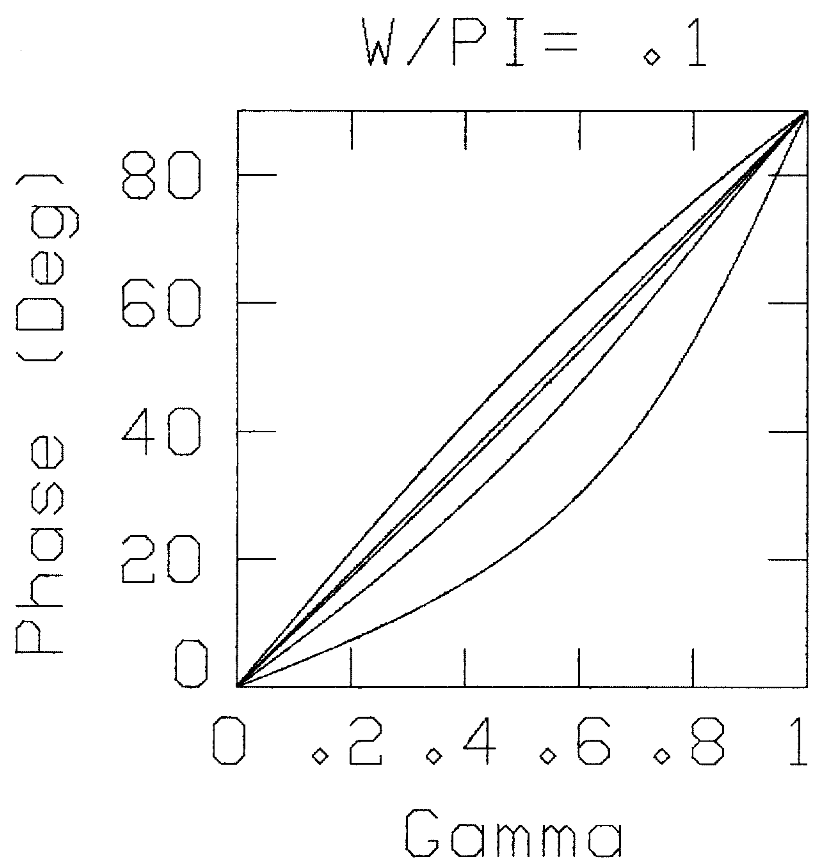
20
40
60
80
0
.2
.4
.6
.8
1
Gamma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Phase (Deg)
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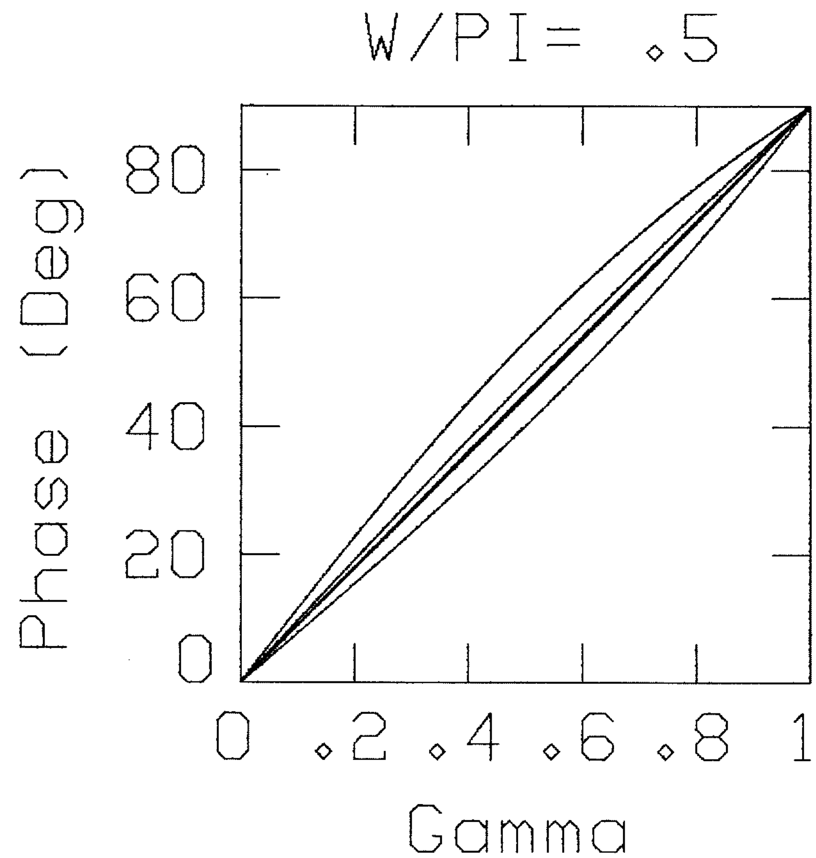
20
40
60
80
0
.2
.4
.6
.8
1
Gamma

FIGURE 2A.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Phase (Deg)
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20
40
60
80
0
.2
.4
.6
.8
1
Gamma

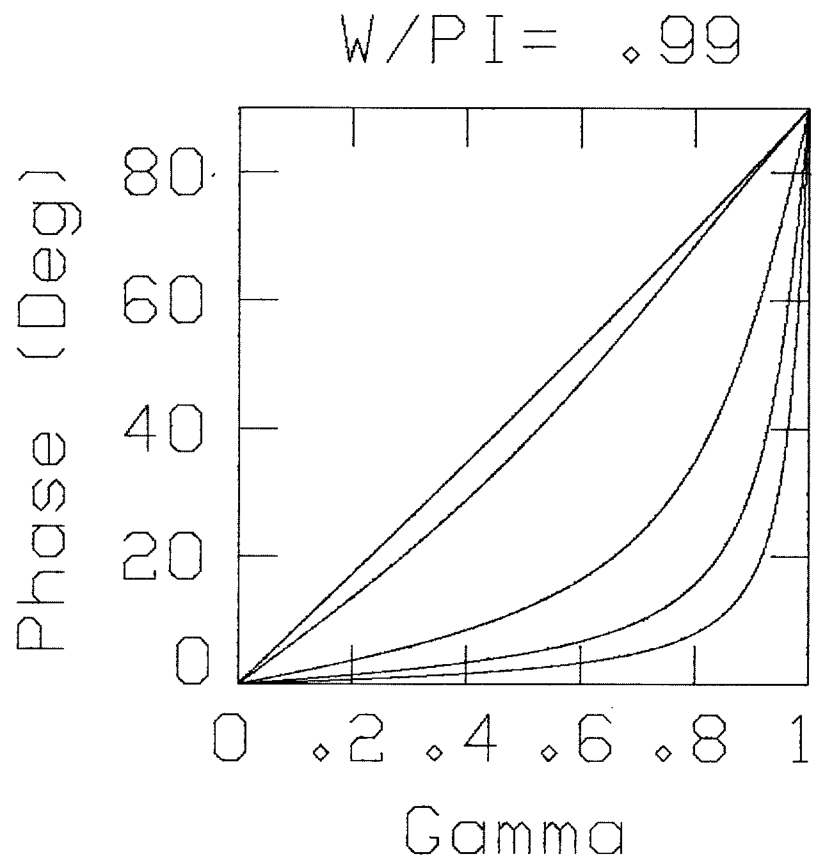
W/PI= .99
Phase (Deg)
0
20
40
60
80
0
.2
.4
.6
.8
1
Gamma

FIGURE 2B.