CMP SEPARATIONS OF THE DOUBLE SQUARE ROOT EQUATION

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The double square root (DSR) operator in common midpoint (CMP) space can be approximately separated into two terms, one involving only migration effects, the other involving only moveout correction. This separation provides an analysis of conventional processing. Estimation of errors in the separation yields an equation that suggests a "devilishly" clever procedure for pre-migration of wide offset data to bring it to zero offset before conventional velocity analysis, stack and final migration. Incorporation of lateral velocity variation exhibits a large, unfamiliar term proportional to the product of the first powers of offset, dip, and lateral velocity gradient.

Fundamentals

The basic ideas spring from the following three definitions of functions of the dip operator $Y \simeq \partial_y^t$ and stepout operator $H \simeq \partial_h^t$, which are themselves developed in "The Double Square Root Equation" (Claerbout, SEP-15, pp. 73-80).

$$DS(Y,H) = [1 - (Y - H)^2]^{1/2} + [1 - (Y + H)]^{1/2} \tag{1}$$

$$Sep(Y,H) = DS(Y,H_0) + DS(Y_0,H) - DS(Y_0,H_0) \tag{2}$$

$$Dev(Y,\hat{H}) = DS(Y,\hat{H}) - Sep(Y,\hat{H}) \tag{3}$$

Equation (1) defines the double square root (DSR) operator, except that the usual scale factor $\pm i\omega/v$ has been omitted. Equation (2) splits this operator into two parts, which are an approximation to (1). For example, if $Y_0 = 0$ and $H_0 = 0$, we have

$$\mathrm{Sep}(Y,H) = 2(1 - Y^2)^{1/2} + 2(1 - H^2)^{1/2} - 2 \tag{4}$$

Notice that if $H = 0$, we have Sep = DS for all Y; likewise, if $Y = 0$, we have Sep = DS for all H. Obviously, if H and Y are both non-zero, the approximation fails to include such terms as H^2Y^2, which would be in a Taylor Series expansion. It is just the absence of such terms, however, that makes (4) *separable*. This means that downward continuation, which roughly amounts to exponentiation of the separation operator, becomes a cascade of two independent operations, one on midpoint and the other on offset. This has many implications for processing history, parameter estimation, cost and accuracy. We often regard the $2(1 - H^2)^{1/2} - 2$ as an NMO-type operation, although what it really does is condense primary information down to zero offset. Stacking amounts to selecting off the zero offset and abandoning the other offsets. Finally, $2(1 - Y^2)^{1/2}$ is the conventional migration of the stack.

In conventional processing the deviation operator $\mathrm{Dev}(Y,\hat{H})$ defined by (3) is customarily ignored. What this operator is trying to do is approximate something like H^2Y^2 without actually requiring the stepout operator $H \simeq \partial_h^t$, which would couple adjoining offsets. H is replaced by some estimate of it $\hat{H}$. Since conventional processing amounts to setting $\hat{H} = 0$ in such terms as H^2Y^2, even crude estimates of $\hat{H}$ may be quite helpful.

Offset stepout prediction

Equations (2) and (3) involve scalar parameters H_0 and $\hat{H}$. How should these be chosen? An appropriate choice might be $H_0 = p\, v(z)$ where Snell's parameter p could be chosen to be, say $\sin 45^\circ/(2.6\ \mathrm{km/sec})$ on the Gulf Coast, and $\sin 45^\circ/(5\ \mathrm{km/sec})$ in the midcontinent.

A more interesting possibility is that H_0 and $\hat{H}$ not be fixed numbers but perhaps variables depending upon the shooting geometry, possibly even functions of offset and traveltime. Let us recall the Fourier solution $\exp(-i\omega t + ik_h h)$, upon which H was defined. Staying at a constant phase implies that $0 = -\omega\, dt + k_h\, dh$ or $dt/dh = k_h/\omega$. Looking at a common midpoint gather we expect to see different stepouts dt/dh at different offsets and times. For a constant velocity medium we can quickly determine that

$$\frac{\partial}{\partial h} \{v^2 t^2 = (2z)^2 + (2h)^2\}$$

$$\frac{dt}{dh} = \frac{4h}{v^2 t}$$

Combining these ideas we get

$$\hat{H} = \frac{vk_h}{2\omega} = \frac{v}{2}\frac{dt}{dh} = \frac{2h}{vt} \tag{5}$$

This illustrates the idea that $\hat{H}(h,t)$ is a function of h and t but not ∂_h^t. In more general cases, $\hat{H}$ could be determined by ray tracing from theoretical velocity models or perhaps estimated by smoothing data. The main point to keep in mind is that great accuracy is not required. Conventional processing takes $\hat{H} = 0$, and almost anything is better than that.

The simplest Devilish operator

Using $Y_0 = 0$ and $H_0 = 0$, we find from (3) that

$$\text{Dev} = [1 - (Y - \hat{H})^2]^{1/2} + [1 - (Y + \hat{H})]^{1/2}$$
$$- 2[(1 - Y^2)^{1/2} + (1 - \hat{H}^2)^{1/2} - 1]$$

This result is far more accurate than is actually required. A second order approximation in Y (see A2 in the appendix) is

$$\text{Dev} \approx [1 - (1 - \hat{H}^2)^{-3/2}]\, Y^2 \approx -\frac{3}{2}\hat{H}^2 Y^2 \tag{6}$$

Incorporating the missing $-i\omega/v$ in the DSR as well as the other substitutions, we get

$$\frac{\partial P}{\partial z} = -\frac{3}{2v}\left(\frac{h}{t}\right)^2 \partial^t_{yy} P \tag{7}$$

It is instructive to note the similarity between the coefficients of (7) and (11-3-18) in *Fundamentals of Geophysical Data Processing*. The former is applied before NMO, however, and the latter is applied after.

Shortcut to the total operator

The total separable operator that could be applied is Sep(Y,H) + Dev(Y,$\hat{H}$) where $H_0 = 0$ and $Y_0 = 0$. As a shortcut to the same result we have

$$\text{Sep}(Y,H;H_0=\hat{H},Y_0=0) = \text{Sep}(Y,H;H_0=0,Y_0=0) + \text{Dev}(Y,\hat{H})$$

which can easily be verified by

$$\begin{aligned} DS(Y,\hat{H}) + DS(0,H) - DS(0,\hat{H}) &= DS(Y,0) + DS(0,H) - DS(0,0) + DS(Y,\hat{H}) \\ &\quad - [DS(Y,0) + DS(0,\hat{H}) - DS(0,0)] \end{aligned}$$

Minimal prestack migration operator

Even in the flattest areas of the world the offset angle H is large. When shooting off-end it would seem to be advantageous for reasons of accuracy and economy to choose both a non-zero $H_0 = pv$ and a non-zero $\hat{H}(h,t)$. Carrying through the algebra we discover that (7) would take the form

$$\frac{\partial P}{\partial z} = \frac{3v}{8}(p^2v^2 - \hat{H}^2)\,\partial^t_{yy} P \tag{8}$$

The advantage of (8) over (7) is that the coefficient is smaller. The fact that computational effort is reduced may not be so important as that the operation does very little to the data, making it more robust. Intuitively, we may regard the prestack operator given by (8) as dip compensating, not to zero offset, but to the offset where the angle pv is achieved.[1] A disadvantage of (8) is that it may be awkward to implement. It cannot be implemented in the ω domain because of the coefficient h/t. It is awkward in the time domain since the coefficient changes sign, thereby changing causality considerations mid-trace. Perhaps it could be implemented in the "radial trace" domain.

Lateral velocity variation

For the study of lateral velocity variation it is convenient to make some slightly different definitions of the basic operators. Let

$$M_s = \frac{1}{v(s)^2} \; ; \qquad M_g = \frac{1}{v(g)^2} \tag{9a,b}$$

$$Y = \frac{k_y}{2\omega} \; ; \qquad H = \frac{k_h}{2\omega} \tag{10a,b}$$

$$H_0 = p \; ; \qquad \hat{H} = \frac{2h}{v^2 t} \tag{11a,b}$$

$$DS = [M_s - (Y - H)^2]^{1/2} + [M_g - (Y + H)^2]^{1/2} \tag{12}$$

With the double square root now explicitly defined for laterally variable media, we get Sep defined by (2) using the shortcut $H_0 = \hat{H}$ and $Y_0 = 0$:

$$\begin{aligned} Sep = \; & [M_s - (Y - \hat{H})^2]^{1/2} + [M_g - (Y + \hat{H})^2]^{1/2} \\ & + (M_s - H^2)^{1/2} + (M_g - H^2)^{1/2} - (M_s - \hat{H}^2)^{1/2} \\ & - (M_g - \hat{H}^2)^{1/2} \end{aligned} \tag{13}$$

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[1] In this respect, Equation (8) is a generalized form of a process similar to the one described by Judson et al (1978). We therefore considered it a proper choice to call the deviation operator "Devilish."

The third and fourth square roots evidently are close to ordinary moveout correction at the average velocity. Abandoning them we will be left with the retarded migration part. Referring to the second order expressions in the appendix, Equation (A3), we recognize that the clutter can be suppressed by the definitions

$$c_s = (M_s - \hat{H}^2)^{1/2} \; ; \qquad c_g = (M_g - \hat{H}^2)^{1/2} \qquad (14a,b)$$

Applying the approximation (A3) to the remaining part of (13), we get

$$\text{Mig} = c_s \left[- \frac{-\hat{H}Y}{c_s^2} - \frac{M_s Y^2}{2c_s^4} \right] + c_g \left[- \frac{\hat{H}Y}{c_g^2} - \frac{M_g Y^2}{2c_g^4} \right]$$

$$\text{Mig} = \hat{H}Y \left[\frac{1}{c_s} - \frac{1}{c_g} \right] - \frac{Y^2}{2} \left[\frac{M_s}{c_s^3} + \frac{M_g}{c_g^3} \right] \qquad (15)$$

The rightmost term is almost identical to the usual migration in laterally homogeneous media. But the first term is entirely new, unfamiliar, and apparently large, namely

$$\text{NewMig} = \hat{H}Y \left[\frac{1}{c_s} - \frac{1}{c_g} \right] \qquad (16)$$

To get more understanding of this term, let us take $\hat{H}$ small and make all the relevant substitutions to obtain

$$P_z = - \frac{h}{t} \frac{v(s) - v(g)}{v^2} P_y \qquad (17)$$

This effect seems to be quite large. If we say that the offset angle h/t is generally of moderate size, then the effect of this equation is in proportion to the two small quantities, dip and lateral velocity gradient, which can be compared to the usual migration effect going as dip squared.

The meaning of this equation can be interpreted for a point scatterer in a medium of lateral velocity gradient. It says that if the top of the diffraction hyperbola on a constant offset (h_1) section is located under midpoint y_1, then for a different section (offset h_2), the top will be located at a different midpoint y_2.

ACKNOWLEDGMENT

The Devilish ® process first came to our attention at the Digicon display booth at the 1977 Calgary SEG meeting. Originally we regarded it as a splitting of *Fundamentals of Geophysical Data Processing* equation (11-3-18). That equation has some severe limitations (constant velocity, 15-degree) as well as a difficult conceptual basis. We puzzled a long time before developing the present simple and accurate mathematical framework. Congratulations to the inventor of Devilish, John Sherwood!

REFERENCE

JUDSON, D. R., P. S. SCHULTZ and J. W. C. SHERWOOD (1978), "Equalizing the Stacking Velocities of Dipping Events via DEVILISH," paper presented at the 48th Annual International SEG Meeting, October 1978, San Francisco.

APPENDIX: SECOND ORDER SQUARE ROOT EXPANSIONS

To second order in ε:

$$(1 - \varepsilon)^{1/2} = 1 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} \tag{A1}$$

To second order in A, i.e. $1 > a >> A$:

$$[1 - (a + A)^2]^{1/2} = (1 - a^2)^{1/2} \left[1 - \frac{aA}{(1 - a^2)} - \frac{A^2}{2(1 - a^2)^2} \right] \tag{A2}$$

To second order in A, i.e. $b > a >> A^2$:

$$[b^2 - (a + A)^2]^{1/2} = (b^2 - a^2)^{1/2} \left[1 - \frac{aA}{b^2 - a^2} - \frac{b^2A^2}{2(b^2 - a^2)^2} \right] \tag{A3}$$

The approximation (A1) is found in all the books. Approximation (A2) may be derived from (A1) by being wary that $a \approx 1$ so that $(1 - a^2)$ is factored from the square root before expansion. Approximation (A3) is a rescaling of (A2).