SPLITTING AND SEPARATION OF DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO THREE-DIMENSIONAL MIGRATION AND LATERAL VELOCITY VARIATION

David Brown

Recently, while Einar Kjartansson, Bert Jacobs and I were working on a 45-degree equation forward modelling problem, we discovered an interesting way to "split" the 45-degree differential operator into two parts - one that corresponds to a purely propagating or "shifting" term (much like time retardation) and one that models only the diffraction effects of the operator. The splitting process is very much like the time retardation process that has been used for many years, but it has the advantage of giving uncomplicated equations even with laterally varying velocity functions. This suggested a very convenient method for doing either migration or diffraction with the 45-degree equation in regions with lateral velocity variation. The method is derived in an example in this paper and has been applied and discussed by Einar Kjartansson in another SEP-15 paper.

With some further effort and with the aid of helpful suggestions by Francis Muir, the splitting technique was generalized in a way that makes analysis of splitting techniques quite straight-forward, and which, in addition, sheds some light on the more general problem of separating differential equations into several parts, each of which may be solved separately. One example that we will consider is the separation of the three-dimensional 15-degree migration process into two separate two-dimensional processes, each of which is applied separately to the data. This has important implications for making three-dimensional migration an economically attractive processing technique.

1. *The theory*

"Splitting" is a technique that was first used as a method for simplifying the numerical approximation of differential equations in many space-dimensions. The idea was to reduce, say, a two-dimensional problem to an alternating sequence of one-dimensional problems, which would then be both cheaper and easier to code than the original problem. A simple example is given in *Fundamentals of Geophysical Data Processing* (Claerbout, 1976), on page 186. Difference methods are discussed in both Richtmyer and Morton (1967, Sections 8.7-8.8) and in Mitchell (1969, Sections 2.12-2.15).

In this paper we will consider the "splitting" or separation of the following problem:

$$[\partial_z - A(x,y,z,\partial_x,\partial_y,\partial_t) - B(x,y,z,\partial_x,\partial_y,\partial_t)]\, u = 0, \tag{1}$$

$$-\infty < t,x,y < \infty$$

with "initial" values $u(x,y,t,z=0) = u_0(x,y,t)$. Here we see that A and B are differential operators that may depend on the operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial t$, and the spatial variables x, y and z. In particular, they do *not* depend on $\partial/\partial z$. Also, the problem is a pure initial-value problem, i.e. there are no boundary conditions.

The solution to this problem at depth z_0 may be written as

$$u(x,y,t,z_0) = e^{\int_0^{z_0} dz\,(A+B)}\, u_0(x,y,t) \tag{2}$$

where the exponentiation of an operator $\int dz\, D$ is defined by

$$\exp\left(\int_0^z dz\, D\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^z dz\, D\right)^n \tag{3}$$

(Kato, 1976, p. 30), and is also an operator.

Now we consider a similar problem in which we have separated the differential Equation (1) into two parts:

$$(\partial_z - A)\,\tilde{u} = 0 \tag{4}$$

with initial conditions $\tilde{u}(x,y,t,0) = u_0(x,y,t)$, and

$$(\partial_z - B)\,\tilde{\tilde{u}} = 0 \tag{5}$$

with initial conditions $\tilde{\tilde{u}}(x,y,t,0) = \tilde{u}(x,y,t,z_0)$, i.e. the solution to Equation (4) at depth z_0. We integrate Equation (5) until depth z_0 and and call the resulting solution $\tilde{\tilde{u}}(x,y,t,z_0)$. The question now is, when will $\tilde{\tilde{u}}(x,y,t,z_0)$ be the same as $u(x,y,t,z_0)$ in Equation (2)? To find out the answer, we write the solution to the problem given by Equations (4) and (5) in exponential form:

$$\tilde{\tilde{u}}(x,y,t,z_0) = e^{\int_0^{z_0} dzB}\, e^{\int_0^{z_0} dzA}\, u_0(x,y,t). \tag{6}$$

Comparing Equations (2a) and (6), we see that $\tilde{\tilde{u}}(x,y,t,z_0) = u(x,y,t,z_0)$ *if and only if the operators* A *and* B *commute with each other,* i.e. if $ABu = BAu$ for any function u. This can be proved by expressing Equations (2) and (6) in their series representation and comparing terms. Equation (2) becomes

$$u = \{1 + \int dz(A+B) + \tfrac{1}{2}[\int dz(A+B)]^2 + \cdots\}$$

$$= \{1 + \int dz(A+B) + \tfrac{1}{2}[(\int dzA)^2 + (\int dzA)(\int dzB) + (\int dzB)(\int dzA) + (\int dzB)^2] + \cdots\}\; u_0$$

while Equation (6) gives

$$\tilde{\tilde{u}} = [1 + \int dzB + \tfrac{1}{2}(\int dzB)^2 + \cdots]\,[1 + \int dzA + \tfrac{1}{2}(\int dzA)^2 + \cdots]\; u_0$$

$$= \{1 + \int dz(A+B) + \tfrac{1}{2}[(\int dzA) + 2(\int dzB)(\int dzA) + (\int dzB)^2] + \cdots\}\; u_0$$

Clearly the two expressions can be equivalent only if $(\int dzB)(\int dzA) = (\int dzA)(\int dzB)$. Since neither A nor B depends on ∂_z, this will be true if $AB = BA$.[1]

The process we have described above is not "splitting" but a more generalized form of splitting that we will call *full separation*. The implications of full separation are much broader than those of splitting. From the arguments above we can see that *a fully-separable-initial-value problem can be solved as two independent initial-value problems.*

When the operators A and B do not commute, the problem is not fully separable. As we will shortly discover, however, if the problem can be expressed in the same form as Equation (1), it can be "split." We consider again Equations (1), (4) and (5), with the same initial conditions as before, but this time we solve each problem only from $z=0$ to $z=\Delta z$. The solution to the full problem [Equation (1)] becomes

$$u = \left\{1 + \int_0^{\Delta z} dz(A+B) + \tfrac{1}{2}\left[\int_0^{\Delta z} dz(A+B)\right]^2 + O(\Delta z^3)\right\} u_0$$

while the solution to the separated problem is

$$\tilde{\tilde{u}} = \left\{1 + \int_0^{\Delta z} dz(A+B) + \tfrac{1}{2}\left[\left(\int_0^{\Delta z} dzA\right)^2 + 2\left(\int_0^{\Delta z} dzB\right)\left(\int_0^{\Delta z} dzA\right) + \left(\int_0^{\Delta z} dzB\right)^2\right] + O(\Delta z^3)\right\} u_0$$

The difference in the two solutions is

$$u - \tilde{\tilde{u}} = \tfrac{1}{2}\left(\int_0^{\Delta z} dzA\right)\left(\int_0^{\Delta z} dzB\right) - \tfrac{1}{2}\left(\int_0^{\Delta z} dzB\right)\left(\int_0^{\Delta z} dzA\right) + O(\Delta z^3)$$

which is clearly $O(\Delta z^2)$. So if we think of solving the separated problem

---

[1]Please see the appendix for a more rigorous proof.

repeatedly for steps of only Δz, we will be making an error that is only $O(\Delta z^2)$ at each step. Since typical difference approximations have a truncation error which is locally $O(\Delta z^2)$, we see that such a "splitting" procedure will not decrease the asymptotic convergence of the method. Björn Engquist has pointed out in SEP-8, p. 153, that this result can be improved so as to converge with an asymptotic rate that is $O(\Delta z^3)$ by utilizing a simple trick. Instead of taking a full step a distance Δz with each half of the splitting procedure, we take a half-step with the first equation [Equation (4)], followed by a full step with the second equation [Equation (5)], followed by a half-step with the first equation again. For the special case of A and B independent of z, the resulting solution will be

$$\tilde{u}(\Delta z) = e^{\frac{\Delta z}{2} A} \, e^{\Delta z B} \, e^{\frac{\Delta z}{2} A} \, u_0 \tag{7}$$

Writing this out in its Taylor's series form, we discover that

$$\begin{aligned} \tilde{u}(\Delta z) &= [1 + \Delta z(A+B) + \frac{\Delta z}{2} (A^2+AB+BA+B^2) + O(\Delta z^3)] \\ &= u(\Delta z) + O(\Delta z^3) \end{aligned}$$

so that the local error of the method is improved to be $O(\Delta z^3)$. Recognizing that A commutes with itself, or that

$$e^{\frac{\Delta z}{2} A} \, e^{\frac{\Delta z}{2} A} = e^{\Delta z A}$$

we only modify the original method by starting off and finishing off with a half-z-step instead of a full z-step. Other than the special starting and finishing procedure, the method is the same as before.

We summarize the results of this section as follows:

The initial-value problem for Equation 1 is fully separable if and only if the operators A *and* B *commute. If* A *and* B *do not commute, then a splitting method will work and can be done with an error which is asymptotically* $O(\Delta z^3)$.

Note that it is important that the differential equation under consideration can be written in the form of Equation (1). We will discover in one of the examples that not all differential equations of interest can be written that way. The 45-degree equation in three space-dimensions is one such case.

2. *Applications*

Three-dimensional migration. The first application we will consider is three-dimensional migration in a stratified medium. The velocity is hence only a function of z, and so the square root downward continuation equation[1] can be written

$$P_z = \frac{1}{v(z)} \left(\sqrt{1 - v^2(z) \; (\partial^{tt}_{xx} + \partial^{tt}_{yy})} \right) P_t \qquad (8)$$

where "$\partial^t P$" is shorthand notation for the definite integral

$$\int_{-\infty}^{t} dt \; P \qquad \text{for a causal process}$$

$$\text{or} \quad \int_{+\infty}^{t} dt \; P \qquad \text{for an anticausal process}$$

The 15-degree approximation to Equation (7) is given by

$$(\partial_z - \frac{1}{v(z)} \partial_t + \frac{v(z)}{2} \partial^t_{xx} + \frac{v(z)}{2} \partial^t_{yy}) \; P = 0 \qquad (9)$$

We recognize that this is in the form of Equation (1) with

$$A = \frac{1}{2v(z)} \partial_t - \frac{v(z)}{2} \partial^t_{xx} \qquad \text{and} \qquad B = \frac{1}{2v(z)} \partial_t - \frac{v(z)}{2} \partial^t_{xx}$$

Since the velocity v is only a function of z, the two operators commute, and hence *the 15-degree three-dimensional migration equation is fully separable in a stratified medium.* This means that three-dimensional data can be downward-continued by first downward-continuing the "inline" sections with

---

[1] In this equation as in all one-way equations we use for migration, we have neglected terms which represent transmission coefficient effects in the z-direction. More information about these missing terms can be gleaned from various articles by Engquist and Brown in SEP-13 and SEP-14.

the 15-degree equation, regrouping the downward-continued data into "crossline" sections and then downward-continuing the crossline sections. The result obtained will be fully equivalent to simultaneous downward continuation in both directions.

The 45-degree equation in three dimensions does not fit the same pattern, however. The 45-degree equation for a stratified earth can be written:

$$\left[\partial_z + \frac{\frac{3v(z)}{4}\left(\partial^t_{xx} + \partial^t_{yy}\right) - \frac{1}{v(z)}\partial_t}{1 - \frac{v^2(z)}{4}\left(\partial^{tt}_{xx} + \partial^{tt}_{yy}\right)} \right] P = 0 \qquad (10)$$

This equation is not in the form of Equation (1) since the denominator of the differential operator contains both x- and y-derivatives. This means that the three-dimensional 45-degree equation cannot be split in the same way that we have been discussing. Instead, we will have to take another approach. We begin with Equation (7), the square root equation in three dimensions. Before approximating the square root in terms of the continued fraction expansions of Muir (see articles by Engquist and Claerbout in SEP-8), we approximate the square root of a sum by a sum of square roots to get the following equation:

$$\left[\partial_z + \frac{1}{v}\partial_t \left(1 - \sqrt{1 - v^2\partial^{tt}_{xx}} - \sqrt{1 - v^2\partial^{tt}_{yy}} \right) \right] P = 0 \qquad (11)$$

It is easy to see that when $\partial_{xx}P = 0$ or $\partial_{yy}P = 0$, Equation (10) reduces to the correct equation. So energy traveling in the inline or crossline direction is treated properly. Energy traveling at offline angles other than 90° will not be treated the same way by Equation (11) as by Equation (8), however. We can get some understanding of the worst case approximation by comparing the dispersion relations of Equations (8) and (11). The dispersion relation of Equation (8) is a perfect half-sphere in (k_x, k_y, k_z)-space, while the dispersion relation for Equation (11) is given by

$$\frac{vk_z}{\omega} = 1 - \sqrt{1 - \frac{v^2 k_r^2}{\omega^2}\sin^2\phi} - \sqrt{1 - \frac{v^2 k_r^2}{\omega^2}\cos^2\phi} \qquad (12)$$

where $\phi = \tan^{-1}(k_y/k_x)$ and $k_r^{\,2} = k_x^{\,2} + k_y^{\,2}$. Equation (12) does the worst job of approximating a half-sphere when $\phi = 45^o$. Equation (12) then becomes

$$\frac{vk_z}{\omega} = 1 - 2\sqrt{1 - \frac{v^2 k_r^{\,2}}{2\omega^2}} \tag{13}$$

The important point to make is that for any reasonable approximation of the square root in Equation (13), the dispersion relation is never worse than the paraboloid of revolution corresponding to the dispersion relation for Equation (9), the three-dimensional, 15-degree equation. In fact, it can be shown that the 45-degree approximation to Equation (11) *always* has a dispersion relation which approximates a half-sphere better than does the 15-degree approximation. In a forthcoming paper, Bert Jacobs will discuss the details of this quasi-splitting of the three-dimensional square root equation in much greater detail.

Splitting the 45-degree equation with lateral velocity variation. The transformation of one-way wave equations to retarded-time coordinate frames has traditionally been motivated by the desire to separate most of the propagation or "shifting" effects of the equations from the diffraction effects. We transform to a frame in which energy traveling straight down in the earth would be seen as not moving at all -- the shifting is mostly taken up by the time retardation and only the diffraction effects remain. For a constant or z-variable velocity medium, this coordinate transformation takes the 45-degree equation,

$$(\partial_{tt} - \frac{v^2}{4}\partial_{xx})\, P_z = \frac{1}{v}(\partial_{tt} - \frac{3v^2}{4}\partial_{xx})\, P_t \tag{14}$$

and converts it to a simpler equation,

$$(\partial_{tt} - \frac{v^2}{4} \partial_{xx}) P_z = - \frac{v}{2} P_{xxt} \tag{15}$$

which can be easier to solve because the triple-time derivative has been removed. When the velocity is a function of x as well as z, however, the resulting equation is not so simple. The equation for a laterally inhomogeneous medium becomes

$$(\partial_{tt} - \frac{v^2}{4} \partial_{xx}) P_z = (\frac{1}{v} - \frac{1}{\bar{v}}) P_{ttt} - \frac{v}{2} (\frac{3\bar{v}-v}{2\bar{v}}) P_{xxt} \tag{16}$$

In addition to having the triple-time derivative, which can be computationally expensive, this equation has coefficients which depend both on $v(x,z)$ and a constant or z-variable reference velocity, $\bar{v}$.

We will take instead another approach to the problem. Rather than using a coordinate transformation to separate the shifting and the diffraction effects of the one-way equations, we will use the approach of Section 1 and *separate* the equations into a shifting and a diffracting part. An equation which is purely shifting is

$$(\partial_z - \frac{1}{v(x,z)} \partial_t) P = 0 \tag{17}$$

We can rewrite Equation (14) to be in the form of Equation (1):

$$\left[\partial_z - \left(\frac{1}{v} \frac{\partial_{tt} - \frac{3v^2}{4} \partial_{xx}}{\partial_{tt} - \frac{v^2}{4} \partial_{xx}} \partial_t - \frac{1}{v} \partial_t\right) - \frac{1}{v} \partial_t\right] P = 0 \tag{14'}$$

The term $(1/v)\partial_t$ has been added and then subtracted to make the separation process clearer.

where $\phi = \tan^{-1}(k_y/k_x)$ and $k_r{}^2 = k_x{}^2 + k_y{}^2$. Equation (12) does the worst job of approximating a half-sphere when $\phi = 45^o$. Equation (12) then becomes

$$\frac{vk_z}{\omega} = 1 - 2\sqrt{1 - \frac{v^2 k_r{}^2}{2\omega^2}} \qquad (13)$$

The important point to make is that for any reasonable approximation of the square root in Equation (13), the dispersion relation is never worse than the paraboloid of revolution corresponding to the dispersion relation for Equation (9), the three-dimensional, 15-degree equation. In fact, it can be shown that the 45-degree approximation to Equation (11) *always* has a dispersion relation which approximates a half-sphere better than does the 15-degree approximation. In a forthcoming paper, Bert Jacobs will discuss the details of this quasi-splitting of the three-dimensional square root equation in much greater detail.

Splitting the 45-degree equation with lateral velocity variation. The transformation of one-way wave equations to retarded-time coordinate frames has traditionally been motivated by the desire to separate most of the propagation or "shifting" effects of the equations from the diffraction effects. We transform to a frame in which energy traveling straight down in the earth would be seen as not moving at all -- the shifting is mostly taken up by the time retardation and only the diffraction effects remain. For a constant or z-variable velocity medium, this coordinate transformation takes the 45-degree equation,

$$(\partial_{tt} - \frac{v^2}{4}\partial_{xx})\, P_z = \frac{1}{v}(\partial_{tt} - \frac{3v^2}{4}\partial_{xx})\, P_t \qquad (14)$$

and converts it to a simpler equation,

$$(\partial_{tt} - \frac{v^2}{4}\,\partial_{xx})\,P_z = -\frac{v}{2}\,P_{xxt} \qquad (15)$$

which can be easier to solve because the triple-time derivative has been removed. When the velocity is a function of x as well as z, however, the resulting equation is not so simple. The equation for a laterally inhomogeneous medium becomes

$$(\partial_{tt} - \frac{v^2}{4}\,\partial_{xx})\,P_z = (\frac{1}{v} - \frac{1}{\bar{v}})\,P_{ttt} - \frac{v}{2}\,(\frac{3\bar{v}-v}{2\bar{v}})\,P_{xxt} \qquad (16)$$

In addition to having the triple-time derivative, which can be computationally expensive, this equation has coefficients which depend both on $v(x,z)$ and a constant or z-variable reference velocity, $\bar{v}$.

We will take instead another approach to the problem. Rather than using a coordinate transformation to separate the shifting and the diffraction effects of the one-way equations, we will use the approach of Section 1 and *separate* the equations into a shifting and a diffracting part. An equation which is purely shifting is

$$(\partial_z - \frac{1}{v(x,z)}\,\partial_t)\,P = 0 \qquad (17)$$

We can rewrite Equation (14) to be in the form of Equation (1):

$$\left[\partial_z - \left(\frac{1}{v}\,\frac{\partial_{tt} - \frac{3v^2}{4}\,\partial_{xx}}{\partial_{tt} - \frac{v^2}{4}\,\partial_{xx}}\,\partial_t - \frac{1}{v}\,\partial_t\right) - \frac{1}{v}\,\partial_t\right] P = 0 \qquad (14')$$

The term $(1/v)\partial_t$ has been added and then subtracted to make the separation process clearer.

When we then separate it into two equations like Equations (4) and (5), one part will be

$$\left(\partial_z - \frac{1}{v} \frac{\partial_{tt} - \frac{3v^2}{4} \partial_{xx}}{\partial_{tt} - \frac{v^2}{4} \partial_{xx}} \partial_t + \frac{1}{v} \partial_t \right) P = 0 \tag{18}$$

and the other part will be Equation (17). "Multiplying" through Equation (18) by $\partial_{tt} - (v^2/4)\partial_{xx}$, it becomes

$$\left(\partial_{tt} - \frac{v^2(x,z)}{4} \partial_{xx} \right) P_z = - \frac{v(x,z)}{2} P_{xxt} \tag{19}$$

Equation (19) is identical to Equation (15), the time-retarded 45-degree equation, except that the velocity is now permitted to be a variable function of both x and z instead of just a constant or z-variable function.

We now come to the question of how to use the separated equations. We recall from Section 1 that if the two separated operators commute, then the equations may be solved independently. In fact, we could forget Equation (17) altogether and the process would be equivalent (in fact, exactly the same) to doing the calculation in a retarded frame. The two operators are

$$A = - \frac{v}{2} \frac{\partial_{xxt}}{\partial_{tt} - \frac{v^2}{4} \partial_{xx}} \qquad \text{and} \qquad B = \frac{1}{v} \partial_t$$

We see immediately that A and B commute only if v is a constant or a function of z only. When v depends on the lateral variable, x, as well, then the operators do not commute. The implication of all this is that *we can downward-continue using the 45-degree equation in a region* *with lateral-velocity variation by solving Equations (17) and (19) alternately at each z-step.*

This method turns out to be particularly simple for doing downward-continuation in the frequency domain. Equation (17) then becomes

$$P_z = -\frac{i\omega}{v} P$$

which can be solved explicitly to give

$$P(z+\Delta z) = e^{-i\frac{\omega}{v}\Delta z} P(z)$$

Thus, the shifting half of the splitting method amounts to a single complex multiplication. This is the method used by Einar Kjartansson in this SEP report.

Many people would still like to do their 45-degree migration in a time-retarded coordinate system. For instance, they may have already written complicated migration routines which, of course, take time retardation into account when imaging the final migrated section. We will give next a method for incorporating lateral velocity effects into the time-retarded problem. We begin again with Equation (14'), but we separate the operator on the left side into three parts:

$$A = \frac{1}{v}\frac{\partial_{tt} - \frac{3v^2}{4}\partial_{xx}}{\partial_{tt} - \frac{v^2}{4}\partial_{xx}}\partial_t - \frac{1}{v}\partial_t \quad , \quad \text{a "diffract";}$$

$$B = \left(\frac{1}{v} - \frac{1}{\bar{v}}\right)\partial_t \; , \quad \text{a "differential shift";}$$

and

$$C = \frac{1}{\bar{v}}\partial_t \; , \quad \text{a "constant shift."}$$

In a laterally inhomogeneous medium, where v depends on x, A and B do not commute with each other. The third operator, C, however, depends only on a constant reference velocity $\bar{v}$ and the time-derivative, so it commutes with both A and B. This means we can downward-continue by solving $(\partial_z - A)P$ and $(\partial_z - B)P$ alternately at each z-step but leaving $(\partial_z - C)P$

until the end. We can even leave out this last equation altogether, and this will be equivalent to time-retarding to the constant reference velocity $\bar{v}$.

Solving the second equation,

$$[\partial_z - (\frac{1}{v} - \frac{1}{\bar{v}})\,\partial_t]\,P = 0$$

at each step will just cause each trace to be shifted by a small amount proportional to

$$(\frac{1}{v} - \frac{1}{\bar{v}})$$

This problem has been discussed in earlier SEP reports for other reasons, and some helpful information may be found in an article by Philip Schultz (SEP-7, p. 75).

3. *An example*

To further convince ourselves of the validity of the theory of Section 1, I ran a simple computational example. The heat-flow equation in two space-dimensions is

$$(\partial_t - \partial_{xx} - \partial_{yy})\,u = 0 \qquad (20)$$

This is in the form of Equation (8) with the z-derivative replaced by a time-derivative. The two operators $A = \partial_{xx}$ and $B = \partial_{yy}$ clearly commute. This means that the equation can be separated and the two resulting equations solved independently. We solved Equation (20) using first a Crank-Nicolson splitting method (see Claerbout, 1976, p. 186) and then solving the problem first in one direction all the way and then in the other direction all the way using the Crank-Nicolson method both times. These methods are shown schematically below:

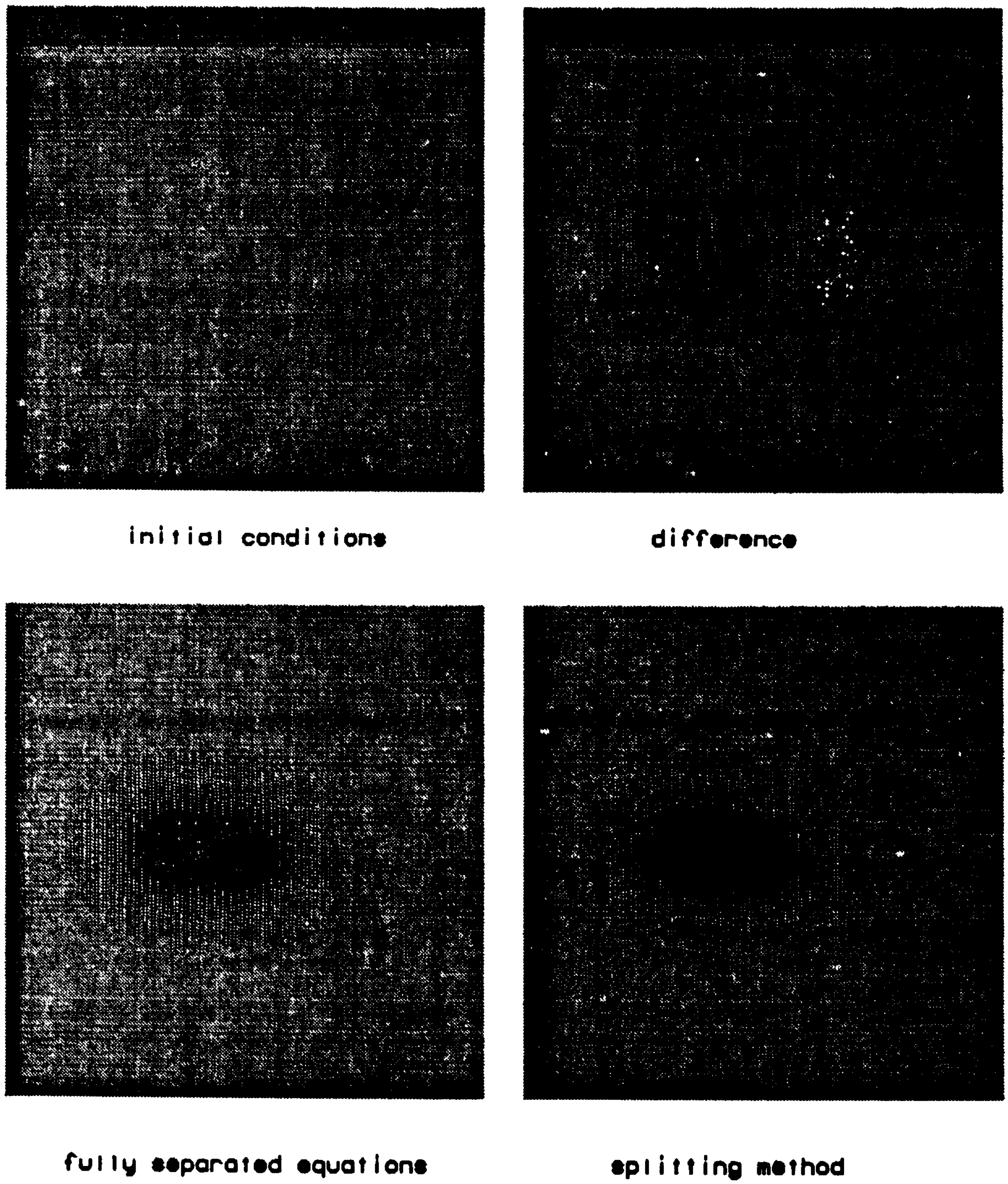

FIGURE 1.--The heat-flow equation was solved using the splitting and separation methods described in the text. Plots of the initial conditions, results, and the difference between the two results are shown here. The scaling factor used when plotting the difference was .8 x 10^{-6} times the scaling factor used with the two lower plots.

Splitting

```
    u(0) = initial conditions
    do 30 i=1,nz
         solve Crank-Nicolson in x
         transpose the result
         solve Crank-Nicolson in y
         transpose the result
30  continue
         plot u(nz)
    end
```

Full-Separation

```
    u(0) = initial conditions
    do 10 i=1,nz
         solve C-N in x
10  continue
    transpose the result
    do 20 i=1,nz
         solve C-N in y
20  continue
    transpose the result
         plot u(nz)
    end
```

The advantage of the full-separation method is apparent -- we have to do 2*nz transposes of the data in the splitting technique, but only two transposes in the separation method. For very large data sets, this can clearly make a massive cost difference.

The results of both methods are shown in Figure 1 for initial data of two nearby instantaneous heat sources. The two results appear to be virtually identical. The difference between the two solutions was also calculated at each point and plotted, but with a different scaling factor; the maximum value of the difference is $.8 \times 10^{-6}$ times the maximum value of the solution. For computational purposes, this is essentially zero. The difference is probably due mainly to the fact that the boundary conditions for the two methods are not exactly equivalent. (Recall that the theory was developed for problems with no boundary conditions. Simultaneous separation of a differential equation and associated boundary conditions would, in general, be much more difficult.)

4. *Pitfalls*

While we have talked about separating equations into two parts, we have so far not said anything about the *stability* of the resulting parts. It is not necessarily true that if we separate a stable equation into two parts, the resulting two equations will also be stable for solving as initial value problems. We still have to check the stability of the new equations as well.

An example in which a stable equation can be split into two equations that are not both stable is the following separation of Equation (16), the time-retarded 45-degree equation, into two parts:

$$(\partial_{tt} - \frac{\bar{v}^2}{4} \partial_{xx}) P_z = -\frac{v}{2} (\frac{3v-\bar{v}}{\bar{v}}) P_{xxt}$$

and

$$(\partial_{tt} - \frac{\bar{v}^2}{4} \partial_{xx}) P_z = (\frac{1}{\bar{v}} - \frac{1}{v}) P_{ttt}$$

(These can be made to look like the form of Equations (4) and (5) by "dividing" through by the operator on the left side of the equations.) The first equation is always stable when Equation (16) is, but the second equation turns out to be *unconditionally unstable* when solved with initial conditions in t and z. In the language of causal positive real operators (see p. 207 this report), this results from the fact that while the sum of two CPR's is always a CPR, the difference is not necessarily a CPR.

APPENDIX

In the proof that the commutivity of A and B is necessary for $e^{zA}\, e^{zB} = e^{z(A+B)}$, we glossed over the details somewhat. We showed clearly why the equality does not hold when A and B do not commute, but we did not show why commutivity necessarily implies that the equality is valid.

If A and B commute, then we can write the binomial expansion:

$$(A+B)^p = \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} A^{p-k} B^k \tag{A1}$$

Now, using the definition for exponentiation of an operator,

$$e^{z(A+B)} = \sum_{p=0}^{\infty} \frac{1}{p!} z^p (A+B)^p$$

$$= \sum_{p=0}^{\infty} \frac{z^p}{p!} \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} A^{p-k} B^k$$

$$= \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{z^p}{k!(p-k)!} A^{p-k} B^k$$

$$= \sum_{p=0}^{\infty} \sum_{n=0}^{\infty} \frac{z^{p+n}}{n!p!} A^p B^n$$

$$= \sum_{p=0}^{\infty} \frac{z^p}{p!} A^p \sum_{n=0}^{\infty} \frac{z^n}{n!} B^n = e^{zA}\, e^{zB}$$

We have used the identity

$$\sum_{p=0}^{\infty} \sum_{n=0}^{\infty} a_p\, b_n = \sum_{p=0}^{\infty} \sum_{k=0}^{p} a_{p-k}\, b_k$$

which holds for any a and b regardless of commutivity.

If A and B do not commute, then the binomial expansion, Equation (A1), becomes

$$(A+B)^p = \sum_{k=0}^{p} \langle A^{p-k}, B^k \rangle \tag{A2}$$

where $\langle A^m, B^n \rangle$ means the sum of all possible combinations of A taken m times and B taken n times and multiplied together. For instance,

$$\langle A^2, B^1 \rangle = A^2B + ABA + BA^2$$

In this case, it is clear that $e^{z(A+B)}$ and $e^{zA}\, e^{zB}$ are not necessarily equivalent.

REFERENCES

CLAERBOUT, J. F. (1976), *Fundamentals of Geophysical Data Processing* (New York: McGraw-Hill).

KATO, T. (1976), *Perturbation Theory for Linear Operators* (New York: Springer-Verlag).

MITCHELL, A. R. (1969), *Computational Methods in Partial Differential Equations* (New York: John Wiley and Sons).

RICHTMYER, R. D. and K. W. MORTON (1967), *Difference Methods for Initial-Value Problems* (New York: Interscience Publishers).