SHORT REVIEW OF RETARDED SNELL MIDPOINT COORDINATES

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A convenient coordinate system in which to formulate many problems of reflection seismic data processing is the retarded, Snell, common midpoint system. It is particularly well-adapted to problems of accurate velocity estimation, migration, and migration before stack when the velocity $v(z)$ is stratified. It also seems to have some utility in the presence of lateral velocity variation, but for this problem and the problem of diffracted multiple reflections, it may be preferable to avoid common midpoint coordinates.

Given the usual definitions of physical variables, the retarded Snell common midpoint coordinate system is defined by:

$$t' = t - p(g-s) + 2\int_0^z \frac{\cos\theta}{v}\,dz \tag{1a}$$

$$y = \frac{g+s}{2} \tag{1b}$$

$$h = \frac{g-s}{2} + \int_0^z \tan\theta\,dz \tag{1c}$$

$$\tau = 2\int_0^z \frac{\cos\theta}{v}\,dz \tag{1d}$$

Before these equations are actually used all of the trigonometric functions are eliminated by Snell's law for stratified media, which says that $\sin[\theta(z)] = pv(z)$, where Snell's parameter p is a numerical constant throughout the analysis. The parameter p characterizes a ray. The coordinate frame (1) can describe any wavefield in any media, but it is particularly advantageous in a stratified media of velocity near $v(z)$ for rays which are roughly parallel to any ray of Snell's parameter p.

Furthermore, at the earth's surface $z = 0$, seismic survey data can be put into this frame merely by numerical choice of p and doing the linear moveout. No knowledge of velocity $v(z)$ is required.

Equation (1d) defines a transformation from depth z to two-way travel-time depth τ. There is an implied vertical speed of $v/\cos\theta$. Because the cosine is in the denominator this is *not* the vertical speed of the tip of the ray. It is instead the speed of vertical motion of the intercept between a wavefront and the z-axis. This speed, known as the vertical phase velocity, turns out to be the appropriate time to depth scaling for Snell waves. Note that the same cosine integral τ in (1d) also appears in (1a) in the definition of retarded time t'. Another term in the definition of retarded time t' is the term $p(g - s)$, which is known as the linear moveout term. Equation (1b) is evidently the usual definition of the midpoint between the shot and the geophone. Equation (1c) defines the surface half-offset h. A peculiar thing about h is that although it agrees with our usual concept of shot to geophone offset at the earth's surface, down inside the earth it is modified by the depth integral of the tangent of the ray angle. Staying at zero offset $g - s = 0$ where the image is found, we see that h increases with depth by this velocity-dependent integral. This fact turns out to provide a velocity determination tool. Recall from the Snell Waves paper the appearance of linearly moved-out common midpoint gathers. The tops of the skewed hyper-boloids are shifted to h values which (for primary reflections) increase with time t'. The location of the tops determines velocity. We will relate $h(z)$ of Equation (1c) to the $h(t')$ for velocity estimation with experimental data by examination of the imaging conditions for migration.

Imaging conditions

Waves can be described in either (g,s,z,t) physical coordinates or the newly defined coordinates (y,h,τ,t'). In physical coordinates we are familiar with the idea that reflectors exist wherever echoes arrive at zero traveltime, namely

$$t = 0 \qquad \text{and} \qquad g = s \tag{2a,b}$$

We would like to express these conditions in the Snell coordinates. Inserting (2) into (1a) and (1d) we get what programmers call the stopping condition

$$t' = \tau \tag{3a}$$

Inserting (2b) into (1c) we have $dh/dz = \tan\theta$ which may be combined with (1d) to give

$$\frac{d\tau}{dh} = \frac{2\cos\theta}{v\tan\theta}$$

Eliminate the trig functions with $pv = \sin\theta$ and solve for v^2. Then eliminate τ with (3a). We get

$$v^2 = \frac{1}{p}\frac{1}{p + \frac{1}{2}\frac{dt'}{dh}} \tag{3b}$$

This equation appeared in the Snell Waves paper as a graphical means of determining the interval velocity from data gathers. Here we have deduced it from (2b), the idea that reflectors are seen by focused energy at zero offset.

Differential equations and Fourier transforms

The chain rule for partial differentiation gives

$$\begin{bmatrix} \partial_t \\ \partial_g \\ \partial_s \\ \partial_z \end{bmatrix} = \begin{bmatrix} t'_t & y_t & h_t & \tau_t \\ t'_g & y_g & h_g & \tau_g \\ t'_s & y_s & h_s & \tau_s \\ t'_z & y_z & h_z & \tau_z \end{bmatrix} \begin{bmatrix} \partial_{t'} \\ \partial_y \\ \partial_h \\ \partial_\tau \end{bmatrix} \tag{4}$$

In our usual notation, time derivative ∂_t has the Fourier representation $-i\omega$. Likewise, $\partial_{t'}$ goes with $-i\omega'$ and the spatial derivatives $(\partial_y, \partial_h, \partial_\tau, \partial_g, \partial_s, \partial_z)$ are associated with $i(k_y, k_h, k_\tau, k_g, k_s, k_z)$. Using these Fourier variables in the vectors of (4) and differentiating (1) to find the indicated elements in the matrix of (4), we have

$$\begin{bmatrix} -\omega \\ k_g \\ k_s \\ k_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -p & \frac{1}{2} & \frac{1}{2} & 0 \\ p & \frac{1}{2} & -\frac{1}{2} & 0 \\ 2\,\frac{\cos\theta}{v} & 0 & \tan\theta & 2\,\frac{\cos\theta}{v} \end{bmatrix} \begin{bmatrix} -\omega' \\ k_y \\ k_h \\ k_\tau \end{bmatrix} \qquad (5a,b,c,d)$$

Let S be the sine of the takeoff angle at the source and G be the sine of the emergent angle at the geophone. If velocity v is known, these angles are directly measurable as stepouts on common geophone gathers and common shot gathers. Likewise, on a constant offset section or a slant stack observed stepouts relate to a sine like quantity Y, and on a linearly moved-out common midpoint gather stepouts measure a sine like quantity H. The precise definitions of these sine-like quantities are given by

$$S = \frac{vk_s}{\omega} \qquad G = \frac{vk_g}{\omega}$$

$$Y = \frac{vk_y}{2\omega} \qquad H = \frac{vk_h}{2\omega} \qquad (6)$$

With these definitions (5b) and (5c) become

$$G = pv + Y + H = Y + (H + pv) \qquad (7a)$$

$$S = -pv + Y - H = Y - (H + pv) \qquad (7b)$$

The case of non-slanted coordinates is seen by substituting $p = 0$ into (7). From (7) we see that one of the effects of linear moveout is to add a shift of pv to H. Setting H equal to zero means setting k_h equal to zero, indicating integration over h, which in turn means slant stacking data with slant angle p. Small values of H/v or k_h/ω obviously refer to stepouts near to p.

Double square root equation

The double square root equation is

$$\frac{k_z}{\omega} = -\frac{1}{v}\left[(1 - S^2)^{1/2} + (1 - G^2)^{1/2}\right]$$

Using the substitutions (5a,d), (6) and (8) we discover that in the retarded Snell coordinates the double square root equation is

$$\frac{k_\tau}{\omega} = 1 - \frac{pv}{1-p^2v^2} H - \frac{1}{2}\left\{\left[1 - \frac{2pv(H-Y) + (H-Y)^2}{1-p^2v^2}\right]^{1/2} + \left[1 - \frac{2pv(H+Y) + (H+Y)^2}{1-p^2v^2}\right]^{1/2}\right\} \tag{8}$$

Application: Velocity estimation in stratified media (Gonzalez)

Here the idea is to collapse hyperboloids on a common midpoint gather not to their tops but to that place on their flanks where they attain some particular stepout p. After this it should be possible to read interval velocities directly as slopes connecting events on the gathers. To gain familiarity with the concept we begin by neglecting dip, that is, insert $Y = 0$ into the double square root equation (8)

$$\frac{k_\tau}{\omega} = 1 - \frac{pv}{1-p^2v^2} H - \left(1 - \frac{2pvH + H^2}{1-p^2v^2}\right)^{1/2} \tag{9}$$

Ordinarily we can migrate a hyperbola top very accurately without knowledge of velocity because the top has no dip and it does not move. Now we want to show that (9) is very insensitive to velocity near stepout = p in (g-s,t) space or equivalently near $H = 0$ in (h,t') space. This means that we can downward continue the hyperbola flank in the neighborhood of stepout p where the velocity information is contained without having to assume that which we are seeking. Expanding (9) in powers of H we see that coefficients of H^0 and H^1 vanish and we get

$$\frac{k_\tau}{\omega} = \frac{H^2}{2(1-p^2v^2)} + 0(H^3) \tag{10}$$

The vanishing of H^0 and H^1 means that data with stepout p do not move as the downward continuation proceeds. After we obtain some experience with (9) upon field data the next step is to incorporate some dip. The next non-zero term in the Taylor Series expansion of (8) about $Y = 0$, $H = 0$ turns out to be

$$\frac{1}{2}\frac{Y^2}{(1 - p^2v^2)^2} \tag{11}$$

Application: Migration of slant stacks (Ottolini)

In non-retarded common midpoint coordinates it is shown elsewhere that downward continuation of slant stacks proceeds with

$$\frac{dP}{dz} = -i\,\frac{\omega}{v}\,\{[1 - (pv + Y)^2]^{1/2} + [1 - (pv - Y)^2]^{1/2}\}\,P$$

Rick Ottolini implemented this with a Stolt-type method (SEP-14, p. 37). To avoid the difficulty of developing a stretch theory along with the mediocre results of stretch methods, he recently implemented the above equation in (z, k_t, ω) space by what we call the telescope method or Gazdag method.

An approach which will offer other advantages is to use retarded Snell coordinates. Inserting $H = 0$ into (8) we obtain

$$\frac{k_\tau}{\omega} = 1 - \frac{1}{2}\left[\left(1 - \frac{-2pvY + Y^2}{1 - p^2v^2}\right)^{1/2} + \left(1 - \frac{2pvY + Y^2}{1 - p^2v^2}\right)^{1/2}\right] \tag{12}$$

At the moment I do not believe we have developed any particularly good wide angle expansions of (12). A splitting method into a sort of S' and G' might be the most effective numerical approach.

Application: Constant offset section migration with velocity estimation

There are some unresolved fundamental difficulties with the idea of exact downward continuation of constant offset sections. Approximate equations are readily found, however. Let us expand the double square root equation (8) in powers of H and Y. Using $(1 - \varepsilon)^{1/2} = 1 - \varepsilon/2$ we get a rather poor approximation, namely

$$\frac{k_\tau}{\omega} = \frac{1}{2}\,\frac{H^2 + Y^2}{1 - p^2v^2}$$

Equation (10) is also a poor approximation. With a little more effort, taking instead $(1 - \varepsilon)^{1/2} = 1 - \varepsilon/2 - \varepsilon^2/8$, and keeping terms to H^2 and Y^2, we obtain the better approximation

$$\frac{k_\tau}{\omega} = \frac{1}{2} \frac{H^2 + Y^2}{(1 - p^2v^2)^2} \tag{13}$$

Because the operators H^2 and Y^2 commute we have *full separation*, a property described elsewhere in this report by David Brown. Essentially it means that the migration part, Y^2, can be done before the velocity analysis part, H^2. Furthermore, if H is applied in the offset domain h instead of the Fourier domain k_h, it means that (13) is a constant offset section migrator. The basic problem that we have with constant offset section migration is that the double square root operator is exactly separable into two terms, one depending only on S, the other only on G, but it is not exactly separable into a term depending on Y plus a term depending on H. Only crude approximations can have this separability. The most accurate but still separable equation that I can devise is the sum of the operators of Equations (9) and (12). This will be called (don't laugh now) the triple square root equation.

$$\frac{k_\tau}{\omega} = 2 - \frac{pv}{1 - p^2v^2} H - \left(1 - \frac{2pvH + H^2}{1 - p^2v^2}\right)^{1/2} - \frac{1}{2}\left[\left(1 - \frac{-2pvY + Y^2}{1 - p^2v^2}\right)^{1/2} + \left(1 - \frac{2pvY + Y^2}{1 - p^2v^2}\right)^{1/2}\right] \tag{14}$$

This equation is valid for all Y when $H = 0$ and for all H when $Y = 0$. In between it errs by cross terms of H^2Y^2 and higher order. Nonetheless it may be excellent in practical work for velocity estimation simultaneous with migration before stack.