DOWNWARD CONTINUING CONSTANT OFFSET SECTIONS
A PARADOX AND FOUR GUESSES

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The downward continuation of shots and geophones proceeds with vertical wave number k_z given by a cosine at the shot and a cosine at the geophone, namely

$$k_z = \frac{\omega}{v} \left[(1-S^2)^{1/2} + (1-G^2)^{1/2} \right] \tag{1a}$$

where G is the sine of the emergent angle, namely vk_g/ω in Fourier Transform space and S likewise relates to the sine of the angle at the shot. It was earlier shown [Migration in Slant-Midpoint Coordinates] that with an angle sine definition for midpoints $Y=vk_y/2\omega$ and one for half-offsets $H=vk_h/2\omega$ that (1a) becomes

$$k_z = \frac{\omega}{v} \left\{ \left[1 - (Y-H)^2 \right]^{1/2} + \left[1 - (Y-H)^2 \right]^{1/2} \right\} \tag{1b}$$

Equation (1) refers to downward continuation of all offsets. The presence of H implies an offset derivative, that is, it implies that a constant offset section with half offset h cannot be downward continued without knowledge of the constant offset section with half offset equal $h + \Delta h$. This is an apparent contradiction to the common knowledge that an impulse in midpoint-traveltime space is known to migrate to an ellipse with one focus at the shotpoint and one at the geophone. A differential equation (11-3-19) in FGDP on page 252 does however purport to downward continue constant offset sections. Tracking down the derivation of the FGDP equation we discover that the offset derivatives have been said to be insignificant by the fact that a normal moveout correction was made. This can be exact for horizontal bedding but it is only an approximation otherwise. In contrast the common sense ellipse solution clearly extends to angles up to ninety degrees. Why does wave theory seem to fail to predict this common sense result? I am guessing that we can achieve the desired ellipsoidal curve by means of an appropriate elimination of H from equation (1b). I will provide a succession of four guesses for the substitution for H . Hopefully the final guess will be valid for all offsets and dip angles.

As we seek to find an exact equation to downward continue the constant offset section we should understand from the beginning that the equation, if we find it, is not likely to be a *local* function of velocity. By this I mean that downward continuation from z_0 to $z_0 + \Delta z$ is not likely to be a function of $v(z_0)$ alone but of $v(z)$ for all z. First let us review why downward continuation of a common midpoint slant stacked section is local and then review the reasons why downward continuation of a constant offset section seemingly cannot be local.

The common midpoint slant stack can be downward continued by a local operation, namely $\partial_z P = ik_z P$ where k_z is given by (1b) in which the slant angle determines a numerical constant value for H and $v=v(z)$ is the local velocity. Migrating a delta function on a constant offset section to an ellipse in a constant velocity medium gives no clue as to whether it is local velocity or global velocity which is required. The approximate procedure of FGDP equation 11-3-19 has small errors if and only if it is possible to do a good normal moveout correction of the data. The consequence is that for even the first 100 meters of downward continuation into the earth in principle we need to know the velocity for all possible depths.

From the point of view of practical migration of primaries in stratified media the non-local property of downward continuations of constant offset sections may or may not be particularly troublesome. However, from the point of view of fundamental theoretical studies to learn how to cope with lateral velocity variation, time-to-depth conversion, statics, diffracted multiples, etc., the extra problems of a non-local downward continuation make the constant offset section an unattractive creature.

Now let us return to the problem of finding a downward continuation equation for constant offset sections in homogeneous media. A check on our proposed equations is whether the wave fronts (in a *group* velocity sense) turn out to be ellipses. In the midpoint-offset fourier domain we have

$$e^{i\Psi} = \exp(-i\omega t + ik_h h + ik_y y)$$

$$= \exp - \frac{i\omega}{v} (tv - 2Hh - 2Yy)$$

Points of stationary phase in (y,h) space are found by setting to zero $\partial_y \Psi$ and $\partial_h \Psi$. Accordingly

$$H = \frac{v}{2} \frac{\partial t}{\partial h} \Big)_y \tag{2a}$$

$$Y = \frac{v}{2} \frac{\partial t}{\partial y} \Big)_h \tag{2b}$$

Our strategy is to replace H in (1b) by something which in some sense approximates H without actually having the need for k_h with its implied coupling of different offsets. The zeroth order guess is to replace H by zero which is to migrate all offsets as if they were vertical stacks (stack without moveout). The first order guess is to think about the flat earth in which the stepout on a common midpoint gather, namely vH , is a predictable function of time t and half-offset h . Given the flat earth travel time equation

$$v^2 t^2 = (2z)^2 + (2h)^2$$

we may differentiate with respect to h at constant z obtaining

$$H = \frac{v}{2} \frac{\partial t}{\partial h} = \frac{2h}{vt} \tag{3}$$

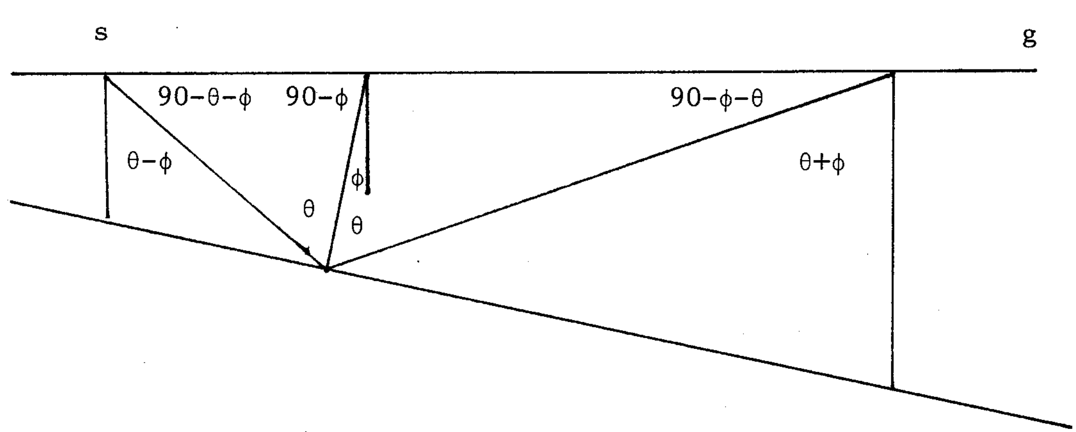
Thus the first order guess is to substitute (3) into (1b). Immediately we recognize another disadvantage of the constant offset section compared to the slanted stack. The slanted stack has a simple numerical value for H in (1b) and the migration, it turns out, can readily be done by a Stolt-like frequency domain method. On the other hand the insertion of (3) into (1b) implies a differential equation with time variable coefficients which makes frequency domain migrations cumbersome and approximate.

The fundamental difficulty with using (3) to predict stepout as a function of offset and travel time is that (3) is not valid in the presence of dipping reflectors. This suggests the second order guess. In a dipping earth the slopes of the hyperboloids will generally be less than those predicted by (3). Experienced seismologists are all familiar with a cosine dip correction to velocity to create the best stacking velocity. Luckily the dip information is contained in the sine like quantity Y . Thus the second order guess is

$$H = \frac{2h}{vt} (1-Y^2)^{1/2} \tag{4}$$

At the time of this writing it is not clear that this guess is not exact, that is, that (4) substituted into (1b) does not give the desired ellipsoidal group velocity curves. Before this test has been made a third guess has been made which hopefully is even more accurate than (4).

First we need Clayton's geometrical construction

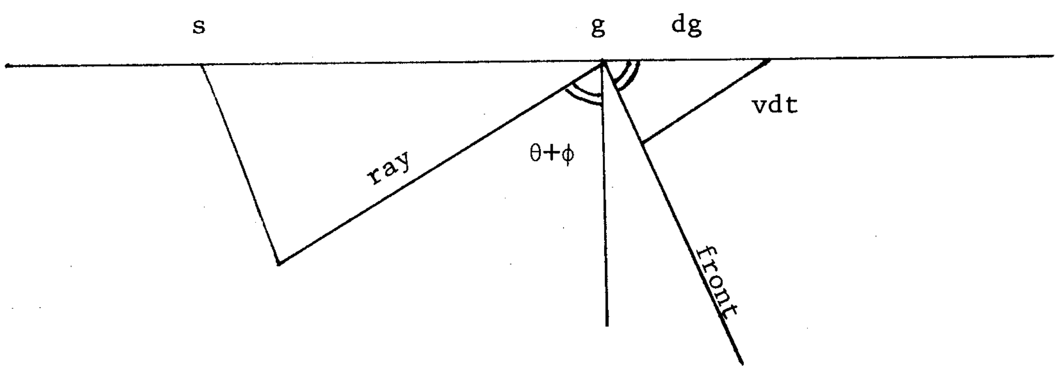


First check that the sums of the interior angles of the triangles are all 180 degrees. Then we have the trigonometric identities

$$\sin(\Theta + \phi) = \sin\Theta \cos\phi + \cos\Theta \sin\phi \quad (=G) \tag{5a}$$

$$\sin(\Theta - \phi) = \sin\Theta \cos\phi - \cos\Theta \sin\phi \quad (=-S) \tag{5b}$$

Consider a plane wavefront with incident angle $\Theta + \phi$ near the geophone.



The travel time increases with g in such a way that

$$G \triangleq v \frac{dt}{dg} = \sin(\Theta + \phi) \tag{6a}$$

Likewise at the shotpoint the travel time decreases as s increases according to the sine of the takeoff angle $\Theta - \phi$

$$S \triangleq v \frac{dt}{ds} = -\sin(\Theta - \phi) \tag{6b}$$

Recall the definitions of shot point and geophone point in terms of midpoint y and half-offset h

$$g = y + h \tag{7a}$$

$$s = y - h \tag{7b}$$

With (6) and (7) equation (2a) becomes

$$H = \frac{v}{2}\frac{\partial t}{\partial h} = \frac{v}{2}\left[\left.\frac{\partial t}{\partial g}\right)_s \frac{\partial g}{\partial h} + \left.\frac{\partial t}{\partial s}\right)_g \frac{\partial s}{\partial h}\right]$$

$$H = \frac{v}{2}\left(\frac{\partial t}{\partial g} - \frac{\partial t}{\partial s}\right)$$

$$H = \frac{1}{2}\ (G - S)$$

$$H = \sin \Theta \cos \phi \tag{8a}$$

Likewise

$$Y = \frac{v}{2}\ \frac{\partial t}{\partial y} = \cos \Theta \sin \phi \tag{8b}$$

What we are basically after is a substitution for H in terms of things like Y, v, h, and t . Solve (8b) for $\sin \phi$

$$\sin \phi = \frac{Y}{\cos \Theta}$$

Insert this into (8a)

$$H = \sin \Theta \left[1 - \left(\frac{Y}{\cos \Theta}\right)^2\right]^{1/2}$$

Finally, associate $\sin \Theta$ with the flat earth guess $2h/vt$. Thus we have the third guess

$$H = \frac{2h}{vt}\left\{1 - Y^2\left[1 - \left(\frac{2h}{vt}\right)^2\right]^{-1}\right\}^{1/2} \tag{9}$$

It now remains to do the computational check that insertion of (9) into (1b) actually does provide a dispersion relation with an ellipsoidal shape.

Epilogue

Rob Clayton has determined that all these guesses do fail to give an ellipsoid. Thus we have not yet won our game of *offset stepout prediction*. Retrospectively it seems to me that the trouble may have originated in equation (2) . The trap may be in what is to be held constant in the differentiation. A better guess than (2) might be

$$H = \frac{v}{2} \left.\frac{\partial t}{\partial h}\right)_Y$$

$$Y = \frac{v}{2} \left.\frac{\partial t}{\partial y}\right)_H$$