VELOCITY ESTIMATION FROM A SINGLE REFLECTOR

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With a given velocity model of the subsurface, rays can be traced from points on the surface, reflected or refracted off layers below, and traced back to the surface. The travel time along each raypath can be computed yielding a set of arrival time curves representing most of the features of the velocity model.

The inverse problem, that of determining a model given the exact time curves, can be solved uniquely if restrictions on the model are assumed. Each year methods are presented which solve for increasingly more complicated models.

A similar inversion problem is to determine the velocity distribution *above* a series of reflectors given the arrival time curves from *only* those reflectors. A solution to this problem has immediate application to statics and velocity estimation as often the reflections in the earlier portions of seismic gathers are completely obscured by noise or multiples. The only information that can be derived from the data about these zones must be obtained from reflections below the noisy zones.

In this paper, inversion theory is applied to a simplified problem of this type in order to examine the properties of the solution. Observations are computed from a horizontally layered model where the depth to the lowest reflector is assumed known. The interval velocities of equally thick layers above are the unknown parameters. The arrival time curve of the lowest layer is inverted to produce an estimate of the interval velocities above, their variances, and the resolution obtainable.

The observations are picked from a seismic gather on which linear moveout corrections have been applied. The moveout correction,

$$T' = T - pX, \qquad (1)$$

is a function of the ray parameter p and the offset X. The reflection is nearly a hyperbola with its top at zero offset in the unshifted coordinate system. In the shifted system, the hyperbola's top moves to some offset X and two-way travel time T'. Claerbout, in SEP-11, shows that the hyperbola's top in the shifted coordinate system is related to the root-mean-square velocity $\overline{v}$ at that offset by

$$\overline{v} = (X/pT')^{1/2}. \tag{2}$$

As the seismic data are coarsely sampled in X, possible errors in picking the hyperbola's top are introduced. These are accounted for by estimating a variance σ_x^2 of the horizontal location. The error in time is assumed insignificant. The variance of the RMS velocity is then

$$\begin{aligned} \sigma_{\overline{v}}^2 &= \sigma_X^2(\partial\overline{v}/\partial X) \\ &= (\sigma_X^2/2pT')(pT'/X)^{1/2}. \end{aligned} \tag{3}$$

A horizontally-layered earth model is assumed. Each layer has a set thickness Z and unknown interval velocity v. The k^{th} observation is related to a model of m layers by

$$X_k = 2\sum_{i=1}^{m} p_k v_i z_i (1 - p_k^2 v_i^2)^{-1/2} \tag{4}$$

and

$$T_k = 2\sum_{i=1}^{m} (z_i/v_i)(1 - p_k^2 v_i^2)^{-1/2} = T_k' + p_k x_k. \tag{5}$$

The partial derivative of the k^{th} RMS velocity with respect to the i^{th} interval velocity can be found from equations (2), (4) and (5):

$$\frac{\partial\overline{v}_k}{\partial v_i} = z_i\left[\left(\frac{p_k}{x_k T_k}\right)^{1/2}(1 - p_k^2 v_i^2)^{-3/2} + \left(\frac{x_k}{p_k T_k}\right)^{1/2}\left(\frac{1}{T_k v_i^2}\right)(1 - p_k^2 v_i^2)^{-1/2}\right]. \tag{6}$$

Given an initial estimate of the model, the objective is to modify that estimate to fit the observations. This is achieved by linearizing the problem using a Taylor expansion and iterating using Newton-Raphson's method in n dimensions until the L_2 norms of successive solutions differ by less than some specified constant.

Using the Taylor expansion, the system of equations relating the observations to the model is

$$[\bar{v}]_{observed} = [\bar{v}]_{model} + [\partial\bar{v}/\partial v][dv] + O(\|dv\|^2). \qquad (7)$$

If n is the number of observations and m the number of layers of the model,

- $[\bar{v}]_{observed}$ is the $n x 1$ vector of RMS velocities observed, computed using equation (2);
- $[\bar{v}]_{model}$ is the $n x 1$ vector of RMS velocities computed using equations (4), (5), and (2);
- $[\partial\bar{v}/\partial v]$ is an $n x m$ matrix relating changes in the computed RMS velocities to changes in the model parameters;
- $[dv]$ is the $m x 1$ vector of changes to be made to the model to achieve a closer fit between the model and observed RMS velocities; and
- $O(\|dv\|^2)$ represents the higher-order terms of the Taylor expansion.

The higher-order terms are assumed insignificant and dropped, yielding a system of linear equations

$$A'x = b', \qquad (8)$$

where

- A' is the $n x m$ partial matrix, $[A']_{k,i} = \partial\bar{v}_k/\partial v_i$;
- x is the $m x 1$ vector representing unknown changes to the parameters of the model: $[x]_{i,1} = dv_i$, where dv_i is the change in the interval velocity of the i^{th} layer required to fit the observations; and
- b' is a $n x 1$ vector representing the difference between the observations and those computed from the model, $[b']_{k,1} = [\bar{v}_k]_{observed} - [\bar{v}_k]_{model}$.

The differing variances of the observations are accounted for by weighting A' and b'. Each row is divided by the standard deviation of the corresponding observation $\sigma_{\bar{v}_k}$. The resulting system,

$$Ax = b, \tag{9}$$

has variances of unity for all observations.

As the thickness of the layers is unknown, and to determine as much about the velocity distribution as the data will permit, a model with more layers than observations is assumed. This leads to an underdetermined system of equations. Such a system has an infinite number of solutions of which there is one that minimizes the changes made to the model. As the problem is nonlinear, this is a very useful criterion, for the Newton-Raphson iteration can become unstable and oscillate if successive solutions vary greatly.

The unique solution is found using a generalized inverse which solves equation (9) such that the norm of the error $||Ax - b||$ and the changes $||x||$ are minimized. The generalized inverse is computed using singular value decomposition discussed by Canales in SEP-10. The decomposition of the partial matrix A is

$$A_{nxm} = U_{nxm} \Lambda_{mxm} V^T_{mxm} , \tag{10}$$

where V is the set of eigenvectors spanning the model space, $V^TV = I$; U is the set of eigenvectors spanning the observation space, $U^TU = I$; and Λ is a diagonal matrix whose elements are the eigenvalues of A.

The least-squares solution to the underdetermined system (9) is

$$x = A^T(AA^T)^{-1}b. \tag{11}$$

Substituting equation (10) into (11), the solution is

$$x = [V \Lambda^{-1} U^T]b. \tag{12}$$

A resolution matrix VV^T is also obtained. Its elements indicate how uniquely the interval velocity in each layer is determined from the observations.

The variance of each interval velocity is related to the decomposition by

$$\sigma_{v_i}^2 = \sum_{j=1}^{n} (v_{ij}/\lambda_j) , \qquad (13)$$

where λ_j is an eigenvalue of Λ. A general paper by Jackson (1972) on inversion theory gives derivations for most of these results.

The reciprocals of the eigenvalues determine the weighting given in the solution to each observation and its variance. If the system is strongly underdetermined, some eigenvalues have very large reciprocals. The variance of the corresponding observation is magnified such that large errors in the solution result. This problem is solved by dropping eigenvectors until the variance of each interval velocity is below some threshold. Dropping the very small eigenvalues reduces the resolution of the interval velocities while keeping their variances reasonable.

Constraints on the solution are imposed by requiring interval velocities to fall within a reasonable range. The upper limit is set or computed from the observations. Each observation is taken for a different value of p. Too high an interval velocity in the model above the reflector would cause the critical angle to be exceeded, making the observation with the highest p impossible. The maximum interval velocity is then

$$v_{max} = 1/p_{max}. \qquad (14)$$

A lower limit is set to the smallest velocity expected.

Because the problem is nonlinear, the solutions given at different iterations generally point in the correct direction, but the distance to move often results in interval velocity estimates outside the acceptable range. When this occurs, the solution can be scaled so the new solution is in the indicated direction but not as far from the previous solution. The scaling is done so that all interval velocities stay within the limits imposed.

Since the constraints are not used directly in the determination of the correction vector, the inversion process often gets trapped at the boundary of one of the constraints. This reflects an incompatibility between one of the constraints and the assumption of a minimum length correction vector. The problem is overcome by adding another degree of freedom to the solution (adding another eigenvalue). This usually points the solution away from the bound but greatly increases its variance.

Another useful constraint is to require velocities for different layers to stay in the same ordering as the initial estimate. Thus, if one starts with an estimate having velocity increasing with depth, all solutions would also have velocity increasing with depth. Linear programming algorithms to accomplish this are possible.

A program was developed to implement the preceding method. It is best summarized by detailing its inputs and outputs.

Inputs:

Initial model:

M	the number of layers of the model
Z	the thicknesses of the layers
V	estimates of interval velocities

Observations:

N	the number of observations
X	horizontal locations of hyperbola tops
T	two-way travel times to tops
VARX	variance of x's
P	ray parameters

Constants:

VMIN	minimum interval velocity acceptable
VARMAX	maximum variance allowed
EPS	threshold used to determine when to stop iterating
IMAX	maximum number of iterations allowed

Outputs consist of plots showing various aspects of the process at each iteration. Figure 1 describes the observations. The dashed lines represent RMS velocities computed from the hyperbola tops at five different offsets. The solid lines show the RMS velocities computed from the model for those offsets. The model starts matching the observations fairly well at about iteration 32.

Figure 2 displays the interval velocities computed at each iteration. The dashed lines represent those used to compute the observations and indicate the correct solution. They are labeled by their thicknesses. Each solid line represents the estimated interval velocity of a hundred-foot depth

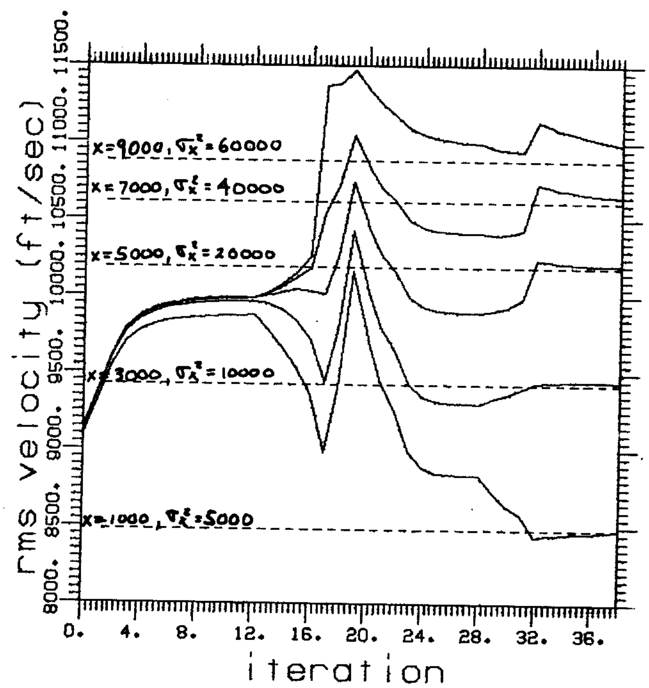

Figure 1.—RMS velocities at different offsets

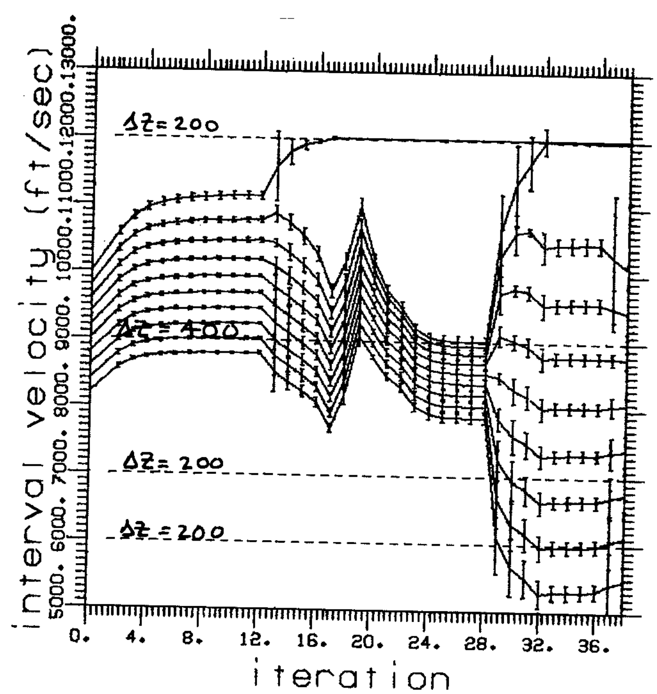

Figure 2.—Interval velocity at different depths.

interval. The vertical bars represent the standard deviation computed from the variances of the observations.

The behavior of the process displayed is typical. The initial estimate, iteration 0, increases until about iteration 16. Up to this point, one eigenvalue has been used to determine some average interval velocity which best fits the observations.

The system is stuck on a solution at this point, so another eigenvalue is used. This enables determination of the velocity of a hundred-foot layer, and of some average value of the rest. A third eigenvalue is used at iteration 30, resulting in the velocity of another hundred-foot layer being estimated correctly.

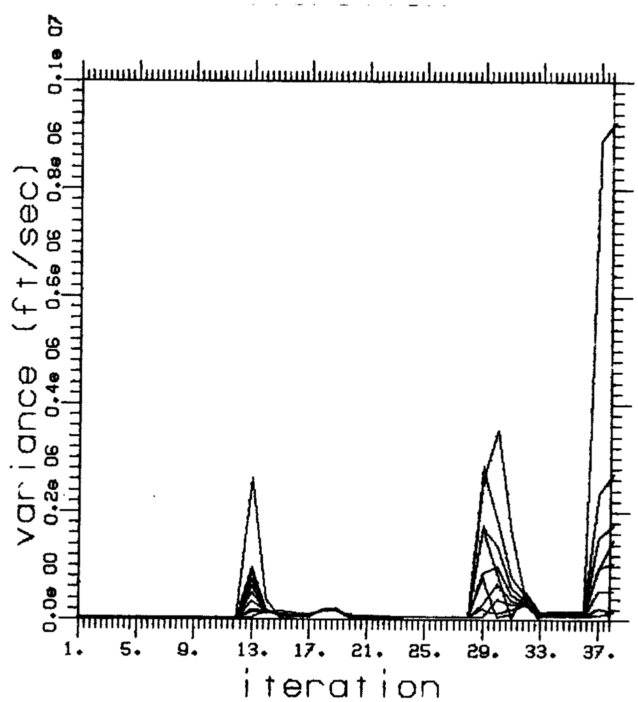

FIGURE 3.—Variance of solution

Figure 3 displays the variance of each component of the solution. It shows the variance increasing as eigenvalues are added but decreasing as new solutions are approached.

Figure 4 shows the logarithms of the eigenvalues. The larger values are related to the 12,000-ft/sec layer as the interval velocity is near critical. For this case, a large change in the observations will cause only a small change in the interval velocity. The other, smaller eigenvalues indicate that small changes in observations cause large changes in the model.

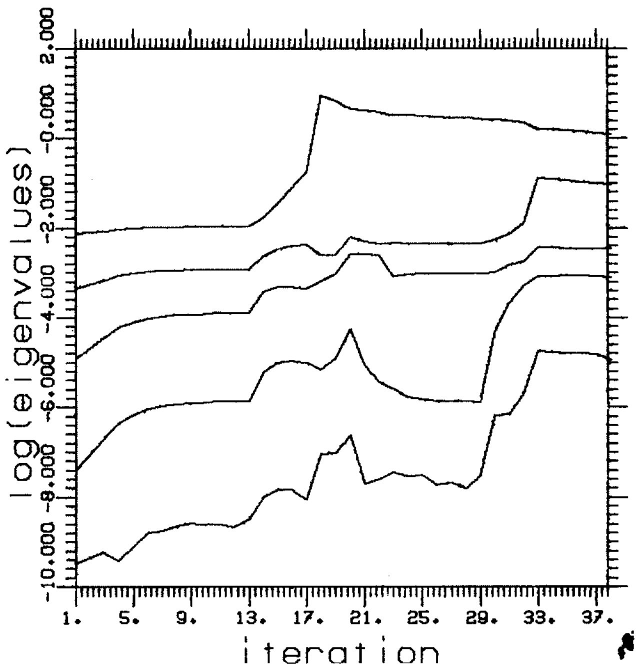

FIGURE 4.—Log_{10} of eigenvalues

The wide range of eigenvalues is typical of a system that is poorly conditioned, that changes to some observation can drastically affect the solution.

Figure 5 shows the number of eigenvalues or degrees of freedom allowed at each iteration. As the system is ill-conditioned, the number of degrees of freedom is restricted to avoid the variance blowing up which would result in a meaningless solution.

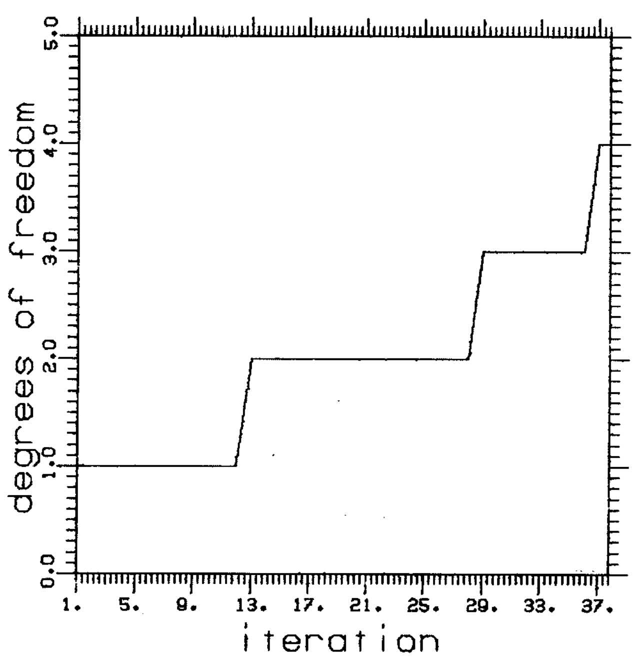

FIGURE 5.—Degrees of freedom

The resolution matrix for the final iteration is plotted to indicate the uniqueness of different components of the solution in Figure 6. If y

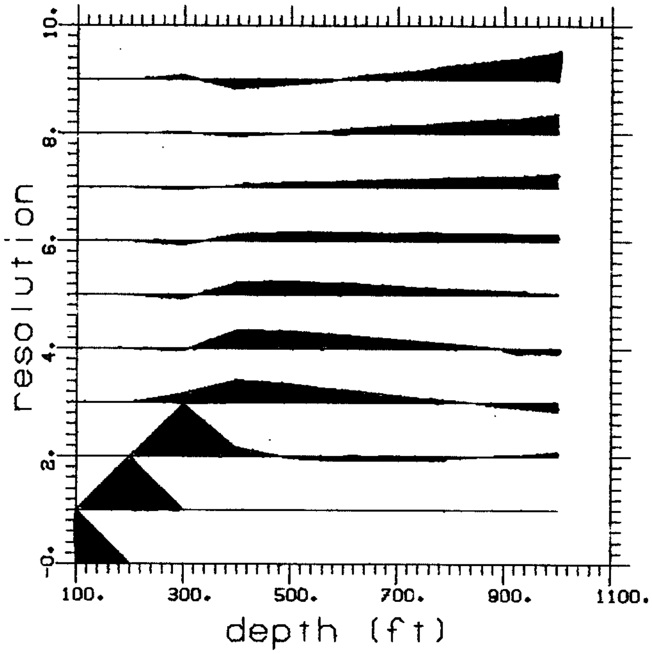

FIGURE 6.—Resolution *versus* depth

is the set of all possible solutions, R the resolution matrix, and x the estimate given by the process, then x = Ry. If R were the identity matrix (which would plot as a set of triangles diagonally up the display), the solution would be unique.

In Figure 6 the shallow high-velocity layers are perfectly resolved, meaning all possible solutions contain a 200-ft, 12,000-ft/sec layer. The velocities for other depths are not well resolved, meaning the solution given is a linear combination of other possible solutions.

Several plots follow which describe the process for variations on the inputs used for the case above. Usually inverting the time curve determines several features of the velocity distribution above with reasonable accuracy. Given enough observations, the process can theoretically produce the correct solution if the velocities are constrained to increase or decrease. In some test cases this happened, but the resulting variances were very large.

Several different inversion methods were programmed, and results are not well enough understood to be presented now. Briefly:

1. Overdetermined systems with small errors were inverted. Generally the solutions oscillated and eventually went unstable.
2. The offsets and times of the arrival time curves were used as observations instead of RMS velocities.

3. Depth and interval velocity were used as unknowns and the time curve inverted. The solutions generally were far from that expected. Layers might have negative depths, often only a few layers resulted, the others having very small thicknesses.

Initially we had hoped to apply the inversion to processed gathers shown in Figure 7. The data were obtained by Mobil Oil on the North Slope. A high-velocity permafrost layer at the surface causes severe Snell's Law effects on the moveout curves. Conventional velocity estimation and stacking smears the reflections as the moveout is not hyperbolic.

Picks were made on the event at 1 sec and a velocity distribution estimated assuming the depth of the permafrost and unfrozen sediments were known. The results could be used in a Snell's Law stacking program to produce a better stacked section.

As the only stable method found so far requires the total depth to the reflector, the method is not so useful. If a stable inversion method can be found that yields both interval velocities and depths, it would have immediate application.

Reference

D. D. Jackson (1972), "Interpretation of Inaccurate, Insufficient and Inconsistent Data," *Geophys. J.* 28, pp. 97-109.

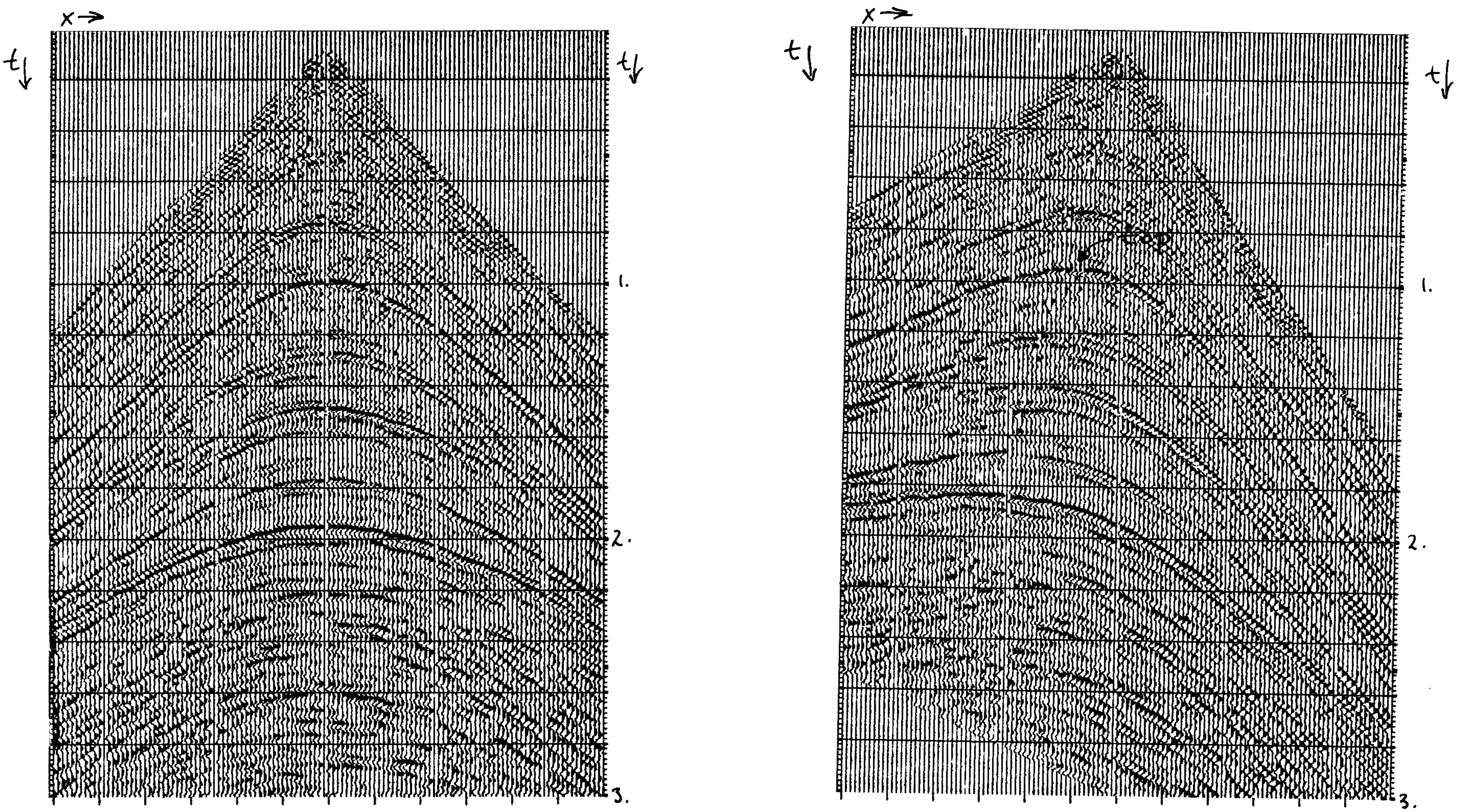

FIGURE 7.—Expanding spread gather. The data on the left were obtained on the North Slope where a high-velocity permafrost layer distorts the moveout of reflections below. Some filtering and a gain function have been applied to the original data. Three seconds of data are shown; the horizontal distance is 13,200 ft. The event at 1 sec was picked for several values of p to obtain observations to invert.

To the right are the same data shifted for $p = \sin 30°/14{,}000$. The top of the hyperbola and its time are picked. Note the large variance ($\sigma_x^2 = 8 \times 10^5$) associated with the offset. The data have little noise and the observations could be picked with a simple algorithm.

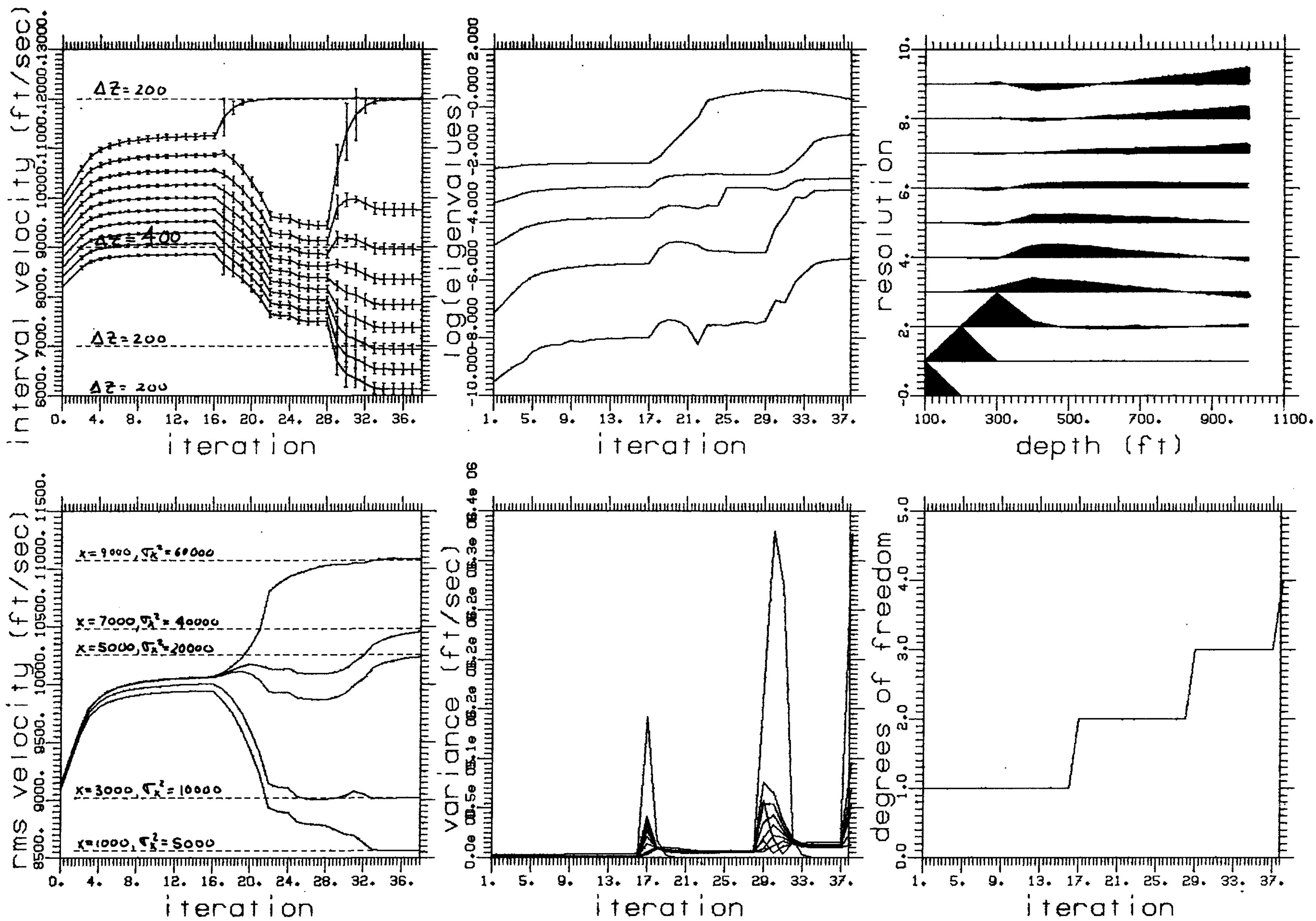

FIGURE 8.—The horizontal picks of the hyperbola tops were varied by ± 25% from their correct values. The process produces almost the same solution as when the data had no errors. Errors in offset only slightly affect the RMS velocities.

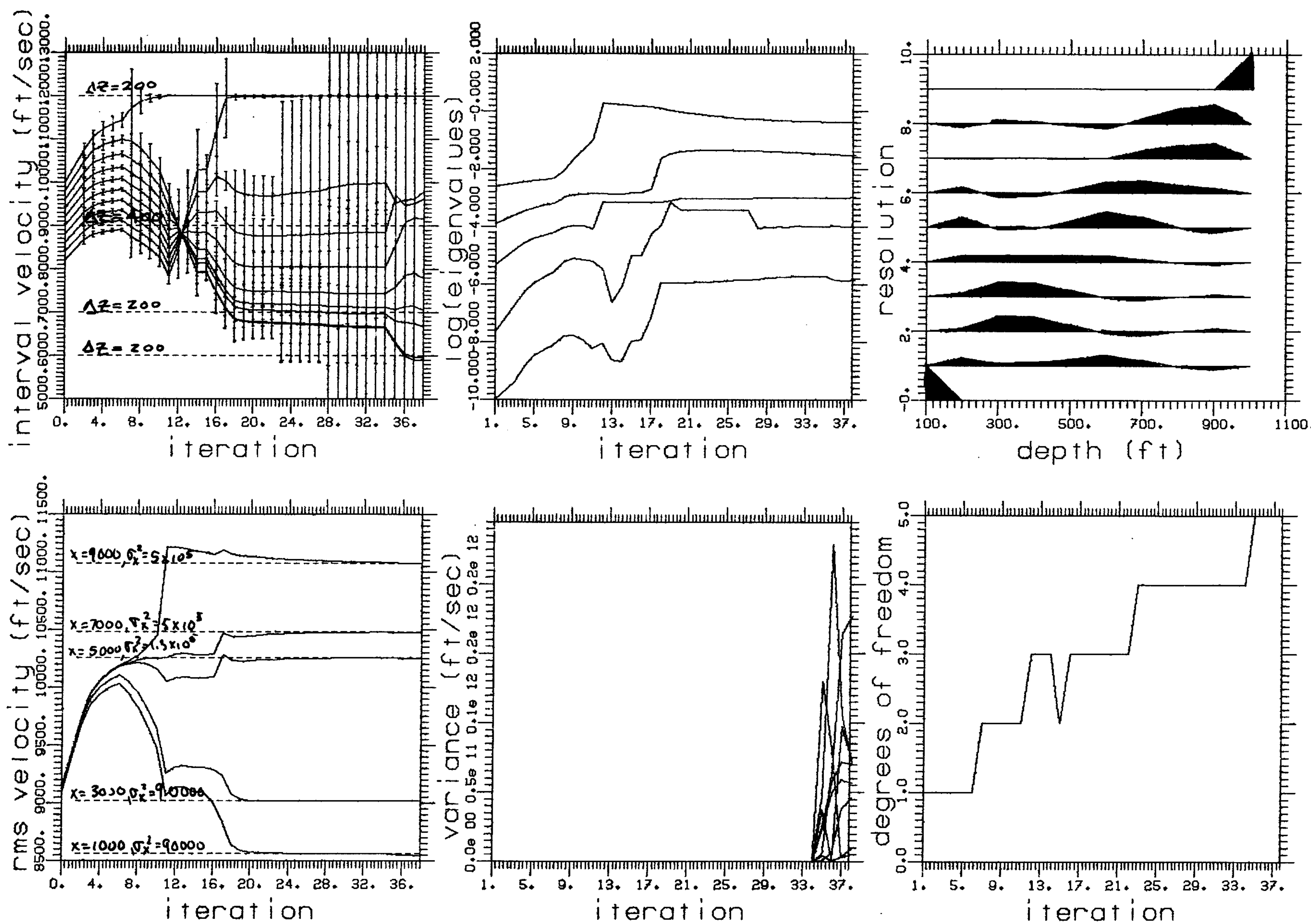

FIGURE 9.—Realistic variances were used with incorrect inputs. The final solution resolves the top and bottom layers correctly, but the variance is very high for the bottom layer.

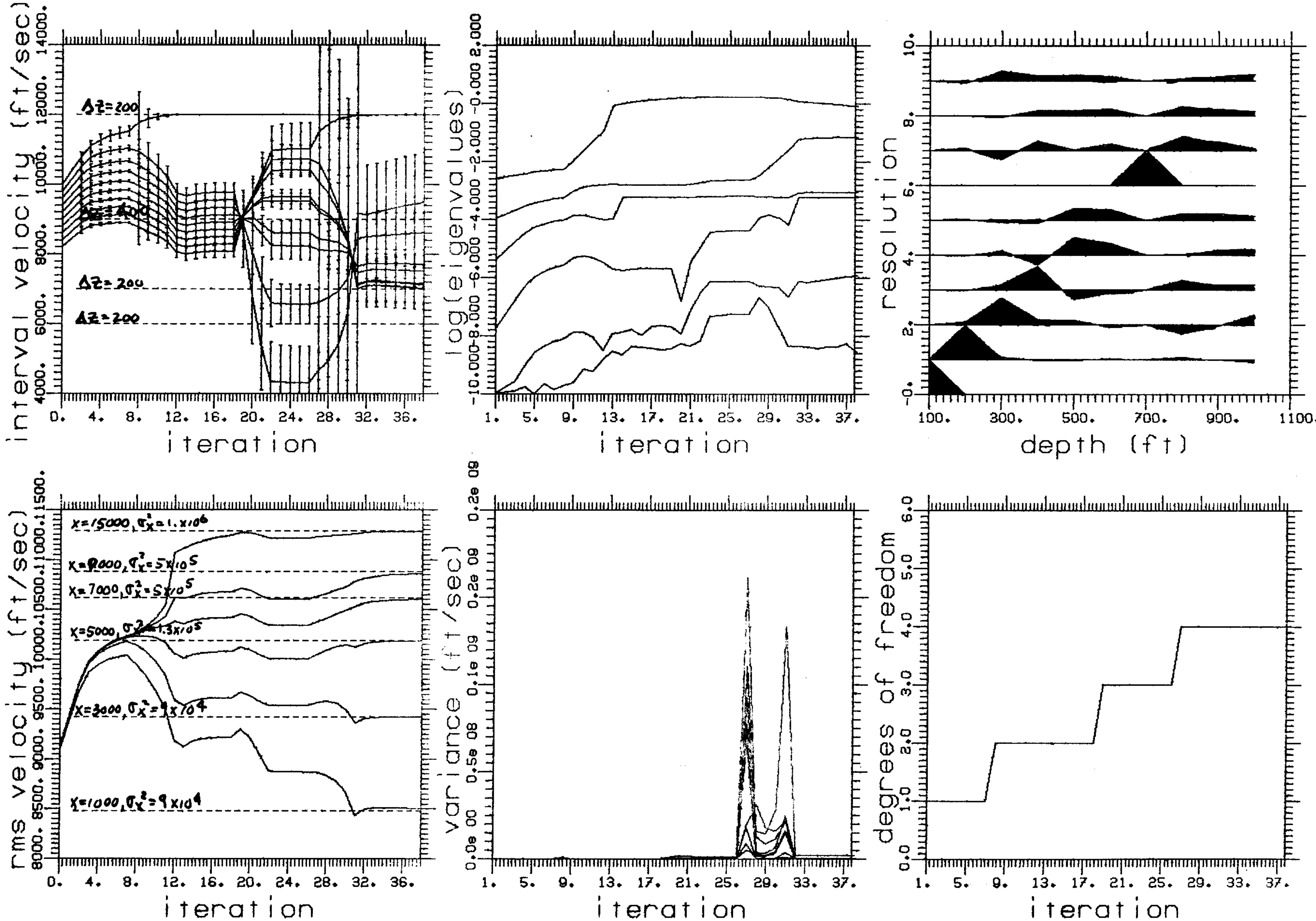

FIGURE 10.—Another observation at 15,000 ft and large variances were input. The velocities at 100, 200 and 700 ft are well resolved