OVERTHRUST IMAGING COORDINATES

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We want to define the principles involved in imaging reflectors with dips of 90° and more (overthrusts). The theory being developed should handle even extreme rays like those depicted in Fig. 1.

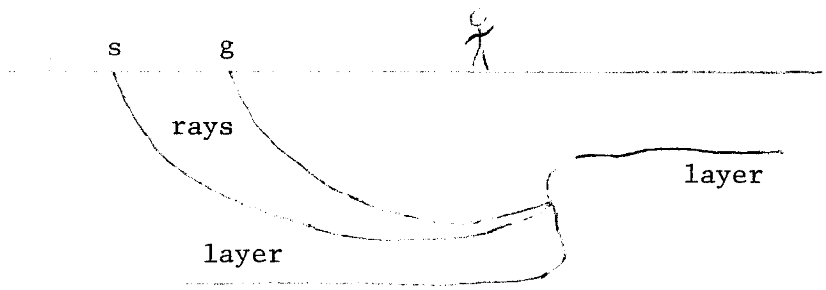

FIGURE 1.—Overthrust imaging.

The coordinate frame we will develop for this problem should be optimum when the waves we are dealing with are generated from a point source moving horizontally along the surface of the earth with any particular velocity $v = 1/p$, where p is Snell's parameter. We usually take v to be increasing with depth. Figure 2 shows a typical ray and a front.

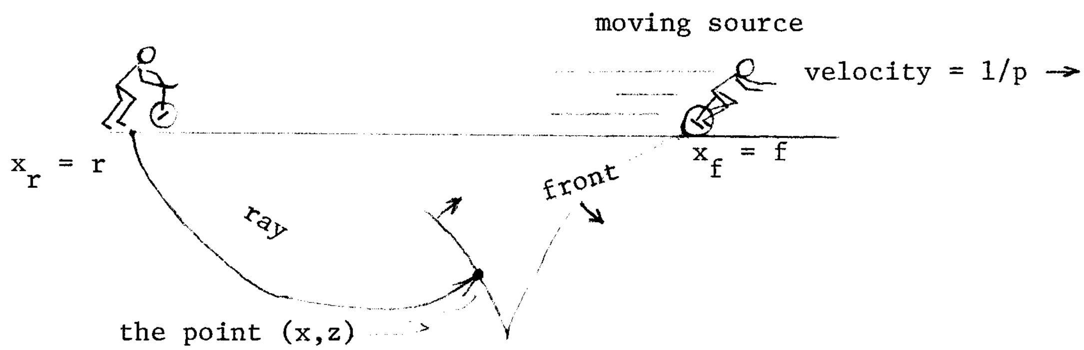

FIGURE 2.—Intersection at (x,z) of a ray and a front. The ray intersects the earth's surface $z = 0$ at $x_r = r$ and the front intersects the earth's surface at $x_f = f$.

The time taken for the source to move along the surface $z = 0$ from $x = r$ to $x = f$ is $p(f - r)$. This is the same time as for the propagation along the ray from the surface at r to the front at the point (x,z). Integrating the horizontal and vertical components of the ray velocity gives

$$x = x_r + \int_0^{p(f-r)} v \sin\theta \, dt , \tag{1a}$$

$$z = \int_0^{p(f-r)} v \cos\theta \, dt . \tag{1b}$$

Equation (1) defines a change of variables from (r,f) to (x,z). This means we can express the wave equation in terms of the independent variables (r,f). Abbreviate $\sin\theta$ by s and $\cos\theta$ by c. By differentiation of (1), we find the Jacobian of the transformation to be

$$\begin{bmatrix} dx \\ dz \end{bmatrix} = \begin{bmatrix} c^2 & s^2 \\ -cs & cs \end{bmatrix} \begin{bmatrix} dr \\ df \end{bmatrix} \tag{2a}$$

$$= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} dr \\ df \end{bmatrix} . \tag{2b}$$

Note that the first matrix is a rotation and the second is a shrinking transformation that is not invertable for $\theta = 0$ and $\theta = 90°$.

It is worthwhile to take a quick look at the constant-velocity case. See Fig. 3.

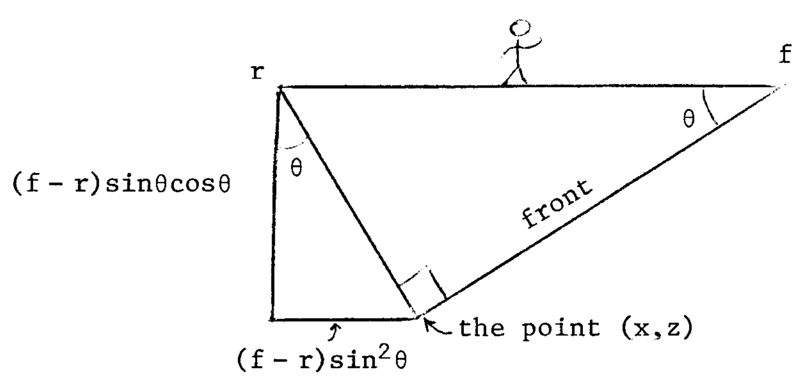

FIGURE 3.—Ray and front intercepting at a point (x,z) in a constant-velocity medium.

From Fig. 3 we get the relations

$$z = (f - r) \sin\theta \cos\theta ,$$
$$x = r + (f - r) \sin^2\theta$$
$$= r \cos^2\theta + f \sin^2\theta,$$

which can be used to confirm the Jacobian (2a). Let us define some primed coordinates by

$$\begin{bmatrix} dx' \\ dz' \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} dr \\ df \end{bmatrix} . \tag{3}$$

Substitution of the right side of (3) into (2b) shows that (dx',dz') is just a rotation of (dx,dz):

$$\begin{bmatrix} dx \\ dz \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} dx' \\ dz' \end{bmatrix} ; \tag{4}$$

hence, dx' and dz' are orthonormal.

Premultiply (4) by the inverse of the rotation matrix

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} dx \\ dz \end{bmatrix} = \begin{bmatrix} dx' \\ dz' \end{bmatrix} . \tag{5}$$

In vertical incidence plane wave studies where $z' = z$, we defined retarded time t' for downgoing waves as $t' = t - \frac{z}{v}$. Now let us define retarded time t' in the locally rotated prime frame by

$$dt' = dt - \frac{dz'}{v} , \tag{6}$$

$$dt = dt' + \frac{dz'}{v} . \tag{7}$$

We now augment (5) with these time variables:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dz \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dt' \\ dx' \\ dz' \end{bmatrix} . \tag{8}$$

Premultiply by the inverse of the right-hand matrix:

$$\begin{bmatrix} 1 & 0 & -1/v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dz \end{bmatrix} = \begin{bmatrix} dt' \\ dx' \\ dz' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} dt' \\ dr \\ df \end{bmatrix} . \tag{9}$$

Premultiply by the inverse of the right-hand matrix and interchange the sides of the equation:

$$\begin{bmatrix} dt' \\ dr \\ df \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1/s \end{bmatrix} \begin{bmatrix} 1 & 0 & -1/v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dz \end{bmatrix} . \tag{10}$$

We have not yet decided whether to solve the wave equation in (t',x',z') coordinates or (t',r,f) coordinates. Performing the usual coordinate transform operations, I have found that the choice is between

$$P'_{t'z'} = \frac{v}{2} P'_{x'x'} , \tag{11}$$

and

$$P'_{t'f} = \frac{v}{2} \frac{\sin\theta}{\cos^2\theta} P'_{rr} . \tag{12}$$

Equation (12) has the disadvantage of an annoying pole at $\theta = \pi/2$. Equation (11) has the advantage of nearly constant coefficients. This means we will probably want to use the orthonormal coordinates (x',z') rather than the surface coordinates (r,f). But there is a catch. Equation (1) provides a global mapping from (r,f) to (x,z). But Eqs. (3) or (4) provides only a local mapping of (dx',dz') to either (dr,df) or (dx,dz).

The (x',z') coordinate frame will involve us with the need to find a numerical technique for injecting energy on a computational mesh at some place other than $z = 0$. Injecting energy at side boundaries has been done by David Brown elsewhere in this report.