MIGRATION WITH FOURIER TRANSFORMS*

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I have recently seen a preprint of a paper by Bob Stolt, of Continental Oil Company, in which he solved the migration problem by Fourier transforms. Since the method could be a serious future competitor for wave-equation migration, I am very interested in the subject and think you will be, too.

Begin with the coordinate transformation used for upcoming waves in wave-equation migration. It is

$$x' = x \ , \tag{1}$$

$$z' = z, \tag{2}$$

$$t' = t + (z/v) \ . \tag{3}$$

Now the chain rule for differentiation of

$$u(x,z,t) = u'(x',z',t') \ ,$$

gives

$$u_z = u'_{z'} z'_z + u'_{t'} t'_z \ ,$$

$$u_z = u'_{z'} + (1/v) u'_{t'} \ . \tag{4}$$

In the Fourier transform domain, we have

$$k'_x = k_x \ , \tag{5}$$

$$\omega' = \omega \ , \tag{6}$$

and

$$ik_z = ik'_z - (i\omega'/v) \ ,$$

$$k'_z = k_z + (\omega/v) \ . \tag{7}$$

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With these preliminaries out of the way, we now take a Fourier transform of the upcoming wave field seen at the surface, i.e.,

$$U(k_x,\omega) = \int dx \int dt \; e^{-ik_x x + i\omega t} \; u(x,z=0,t) \; . \tag{8}$$

Inverse transformation gives (to within a scale factor)

$$u(x,z=0,t) = \int dk_x \int d\omega \; e^{+ik_x x - i\omega t} \; U(k_x,\omega) \; . \tag{9}$$

To propagate waves down a telescope, one uses the transfer function

$$\exp(ik_z z) = \exp\{i[(\omega^2/v^2) - k_x{}^2]^{1/2} \; z\} \; .$$

The only subtle part of this derivation is that for migration we want to continue downward with k_z' given by (7) rather than with k_z, since horizontal events shouldn't migrate. That is, the downward continuation we want is

$$u'(x,z,t') = \int dk_x \int d\omega \; e^{ik_x x - i\omega t'} \; e^{ik_z' z} \; U'(k_x,\omega) \; , \tag{10}$$

where (1), (2), (5), and (6) have been used to eliminate primed coordinates. Using (7), the migrated wavefield is now

$$u'(x,z,\tfrac{z}{v}) = \int dk_x \int d\omega \; e^{ik_x x} \; e^{i[(\omega^2/v^2) - k_x{}^2]^{1/2} \; z} \; U(k_x,\omega) \; . \tag{11}$$

The only trouble with this is that you wouldn't want to have to inverse transform at each possible z level. Luckily, the problem is removed by a change of independent variable from ω to k_z, namely

$$k_z = [(\omega^2/v^2) - k_x{}^2]^{1/2} \; , \tag{12}$$

$$\omega = v(k_x^2 + k_z^2)^{1/2} , \tag{13}$$

$$d\omega/dk_z = \frac{vk_z}{(k_x^2 + k_z^2)^{1/2}} = v \cos\theta \quad . \tag{14}$$

Applying this change of variable to (11), we get Stolt's result:

$$u'(x,z,\frac{z}{v}) = \int dk_x \int dk_z \quad e^{ik_x x + ik_z z}$$

$$\times \quad \frac{vk_z}{(k_x^2 + k_z^2)^{1/2}} U[k_x, v(k_x^2 + k_z^2)^{1/2}] \; . \tag{15}$$

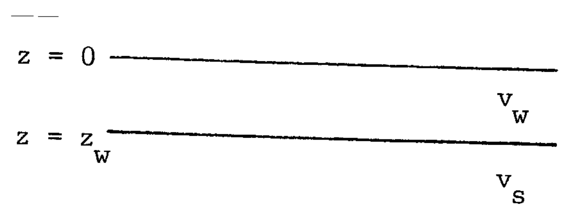
We recognize that this result is in the form of a double Fourier transform.

Now we may wonder how this method will stack up against the wave-equation method. One clear advantage of the FT method is this: it works to all angles and up to the aliasing frequencies in time and space. One clear disadvantage of the FT method is this: it isn't at all obvious how to handle space variable velocity. Particularly troublesome is the usual variation from water velocity of 1.5 km/sec to sediment velocities on the order of 3 km/sec.

To ease this difficulty (and to be sure you have been paying attention to the lecture), I shall assign you a home quiz. Please limit yourself to three hours. The problem is this: Consider a water layer of velocity v_{water} and thickness z_{water} overlying an earth of uniform velocity v_{sed}. Write a closed-form solution in terms of Fourier transforms for the migrated reflectors beneath the water layer. Hint: Be sure your answer goes Stolt's answer when $z_{water} = 0$ or when $v_{water} = v_{sed}$.

SOLUTION

Walt Lynn



The surface data is

$$u(x,z{=}0,t),$$

and its Fourier transform is

$$U(k_x,0,\omega) \;=\; \int dx \;\int dt\; e^{-ik_x x \,+\, i\omega t} \quad .$$

Assuming no reflectors in the water layer, the Fourier transform of the wavefield at $z = z_w$ is

$$U(k_x,z{=}z_w,\omega) \;=\; U(k_x,0,\omega)\; e^{ik_z z_w} \;, \qquad k_z = \left(\frac{\omega^2}{v_w^{\,2}} - k_x^{\,2}\right)^{1/2} \;.$$

Now define a new coordinate system which is translated in z and retarded in time:

$$x' = x \;; \qquad z' = z - z_w \;; \qquad t' = t + (z - z_w)/v_s \;.$$

Let $u'(x',z',t')$ be the wave field described in this coordinate system:

$$u'(x',z'{=}0,t') \;=\; u(x,z_w,t) \;,$$

or

$$U'(k'_x,z'{=}0,\;\omega') \;=\; U(k_x,\,z{=}z_w,\omega)$$

$$= U(k_x,z{=}0,\omega)\; e^{ik_z z_w} \;. \qquad \text{(S-1)}$$

Following the method discussed in the lecture, we want

$$U'(k'_x,z',\omega') \;=\; U'(k'_x,0,\omega')\; e^{ik'_z z'} \;,$$

or

$$u'(x',z',t') = \frac{1}{2\pi}\int dk_x' \int d\omega' \; U'(k_x',0,\omega')\, e^{ik_x'x' - i\omega't'}\, e^{ik_z'z'} ,$$

using (7) and $\omega = \omega'$, $k_x = k_x'$,

$$= \frac{1}{2\pi}\int dk_x' \int d\omega' \; U'(k_x',0,\omega')$$

$$\times\; e^{ik_x'x' - i\omega't'}\, e^{i\{[(\omega'^2/v_s^2) - k_x'^2]^{1/2} + (\omega'/v_s)\}z'} .$$

For a migrated section, we want $t = 0$ or $t' = (z - z_w)/v_s = z'/v_s$:

$$u'(x',z',\frac{z'}{v_s}) = \frac{1}{2\pi}\int dk_x' \int d\omega' \; U'(k_x',0,\omega')\, e^{ik_x'x' - i\omega'(z'/v_s)}$$

$$\times\; e^{i[(\omega'^2/v_s^2) - k_x'^2]^{1/2}z' + i(\omega/v_s)z'} .$$

Simplifying,

$$u'(x',z',\frac{z'}{v_s}) = \frac{1}{2\pi}\int dk_x' \int d\omega' \; U'(k_x',0,\omega')\, e^{ik_x'x'}\, e^{i[(\omega'^2/v_s^2) - k_x'^2]^{1/2}z'} .$$

Change variables to

$$k_z = [(\omega'^2/v_s^2) - k_x'^2]^{1/2} ,$$

$$\omega = v_s(k_x'^2 + k_z^2)^{1/2} , \qquad d\omega/dk_z = v_s\; k_z/(k_x'^2 + k_z^2)^{1/2} ,$$

and

$$u'(x',z',\frac{z'}{v_s}) = \frac{1}{2\pi}\int dk_x' \int dk_z \;\; U'[k_x',0,v_s(k_x'^2 + k_z^2)^{1/2}]$$

$$\times\; [v_s k_z/(k_x'^2 + k_z^2)^{1/2}]\; e^{ik_x'x' + ik_z z'} .$$

Using Eq. (S-1), we can write the solution in terms of the Fourier transform of the surface data:

$$u\left(x,\ z-z_w, \frac{z-z_w}{v_s}\right) = \frac{1}{2\pi}\int dk_x \int dk_z \frac{v_s\,k_z}{(k_x^2 + k_z^2)^{1/2}}\ U[k_x, 0,\ v_s(k_x^2 + k_z^2)^{1/2}]$$

$$\times\ e^{i[(v_s^2/v_w^2)(k_x^2+k_z^2) - k_x^2]^{1/2} z_w}\ e^{-ik_z z_w}$$

$$\times\ e^{ik_z z + ik_x x}\ ;$$

for $z_w = 0$:

$$u(x,z,\frac{z}{v_s}) = \frac{1}{2\pi}\int dk_x \int dk_z \frac{v_s\,k_z}{(k_x^2+k_z^2)^{1/2}}$$

$$\times\ U[k_x,0,v_s(k_x^2+k_z^2)^{1/2}]\ e^{ik_z z+ik_x x}\ ;$$

for $v_w = v_s$:

$$u(x,z,\frac{z}{v_w}) = \frac{1}{2\pi}\int dk_x \int dk_z \frac{v_w\,k_z}{(k_x^2+k_z^2)^{1/2}}$$

$$\times\ U[k_x,0,v_w(k_x^2+k_z^2)^{1/2}]\ e^{ik_z z+ik_x x}\ .$$