A Note on Waveform Estimation

by Luis Canales

We start with equation (3) from "Refined Source Waveform Estimation" (Claerbout, this report, p. 51).

$$U(z) = \frac{B(z)\,R(z)}{B(z)+R(z)} \tag{1}$$

(Note that we have dropped the primes.)

We want to minimize

$$\Sigma\, u_k^{\,2} = P = \text{Noah Power} \tag{2}$$

Let U_k , B_k and R_k be the frequency domain coefficients of u_i, b_i, r_i . Using Parseval's theorem for the discrete fourier transform we get

$$\sum_{k=1}^{N} u_k^{\,2} = \frac{1}{N}\sum_{j=1}^{N} |U_j|^2 = P \tag{3}$$

In the absence of any extra constraint, the conditions to minimize P are to set

$$\frac{\partial P}{\partial B_j} = \frac{1}{N}\sum_{k=1}^{N} U_k \frac{\partial U_k}{\partial B_j} = \frac{1}{N}|U_j|^2 \frac{U_j}{B_j^2} = 0 \tag{4}$$

$$= \frac{B_j^* \,|R_j|^2\, R_j}{|B_j^* + R_j^*|^2 (B_j+R_j)} = 0$$

so that $B_j = 0$ and $B_j = \infty$ are the solutions to the problem.

This shows that in order for the problem to be well posed we need some extra constraints. Claerbout already discussed limiting the degrees of freedom by forcing B to be short. In what follows we propose to force the source's energy to be a constant and we show that the problem becomes well posed. The constraint is then

$$\Sigma\, b_i^{\,2} = Q = \text{Constant}$$

Using again Parseval's theorem and introducing the Lagrange's multiplier λ we need to minimize

$$\frac{1}{N}\left[\sum_{j=1}^{N} |U_j|^2 + \lambda \sum_{k=1}^{M} |B_k|^2\right] = P + \lambda Q \tag{5}$$

As before we have

$$\frac{\partial}{\partial B_k}(P+\lambda Q) = U_k^* \frac{U_k^2}{B_k^2} + \lambda B_k^*$$

$$= \frac{B_k^* R_k^* R_k^2}{|B_k+R_k|^2 (B_k+R_k)} + \lambda B_k^* = 0 \tag{6}$$

We could solve (6) for all frequencies and find b_j by transforming back to the time domain, but this is not easy to do, because the equation is non-linear and because we also would need to adjust the parameter λ until the constraint is correctly satisfied. We can nevertheless see that $B_k = \infty$ is not a solution anymore and $B_k = 0$ for all k is unacceptable because we need $\frac{1}{N}\Sigma\,|B_k|^2 = Q$. So, even though the solution is not easy to get this way, we see that the problem is now well posed.

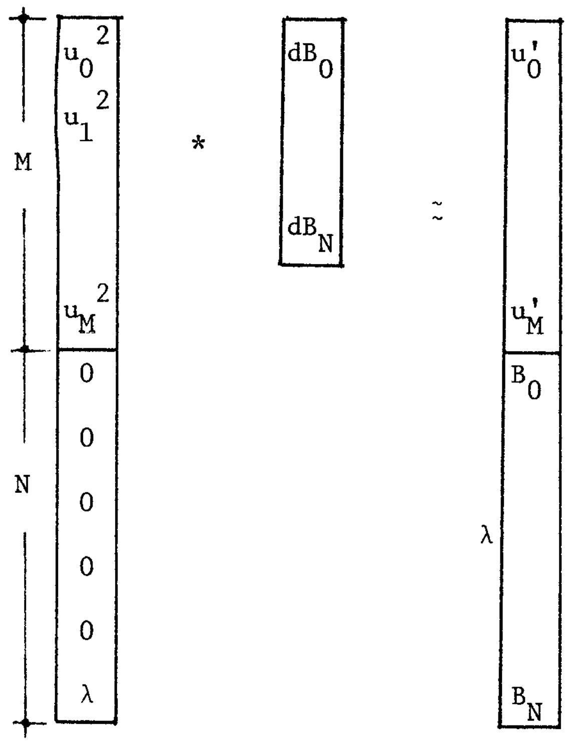
Instead of pursuing the matter any more in the frequency domain, let us incorporate the constraint into Claerbout's main algorithm (equation (8), "Refined Source Waveform Estimation", this report, p. 52). It is then modified to

$$\text{Minimize} \quad \{ \, ||U' + U^2 \, d\,B|| \; + \lambda||B + d\,B \,|| \, \}$$

In matricial notation we have to solve the following least squares (or any other norm) problem.

$$\left[\begin{array}{cccc} u_0^2 & 0 & 0 & 0 \\ u_1^2 & u_0^2 & 0 & 0 \\ u_2^2 & u_1^2 & u_0^2 & 0 \\ u_3^2 & u_2^2 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ \mathbf{0} & & \ddots & u_n^2 \\ \hline \lambda & & \mathbf{0} & \\ & \lambda & & \\ & & \lambda & \\ \mathbf{0} & & \ddots & \lambda \end{array}\right] \left[\begin{array}{c} dB_0 \\ dB_1 \\ dB_2 \\ \vdots \\ dB_m \end{array}\right] \approx \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ -\lambda \\ \\ \end{array} \left[\begin{array}{c} u'_0 \\ u'_1 \\ \vdots \\ u'_n \\ \hline B_0 \\ B_1 \\ B_2 \\ \vdots \\ B_m \end{array}\right]$$

In order to use the algorithm LEVITY we have to transform the arrays u^2 and u' and define the problem as follows:



This is very similar to the damped least squares method of Levenberg and Marquardt. The choice of λ is very subjective, and we should develop an a-posteriori strategy, depending on the results obtained. If we know the power of the source (i.e., Q) the choice of λ should be unique.

Note that if instead of using LEVITY you use Levinson's recursion, the zero-lag value of the autocorrelation is modified to $r_0 := r_0 + \lambda^2$, so that the autocorrelation matrix is always full rank.