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Absorbing Boundary Conditions for Wave Equations

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When calculating solutions to partial differential equations we often need to introduce artificial boundaries to limit the area of computation. This is, for example, the case when we only have measurements in a limited region or when the physical domain is too large for computational purposes.

We need boundary conditions at these artificial boundaries in order to get a unique and bounded solution to the differential equation. Our desire is of course that these extra boundary conditions affect the solution such that it is close to the solution we would have without artificial boundaries. In particular, we would like waves generated in the interior not to be reflected at these boundaries.

It is in general not possible to have well posed problems with fully absorbing local boundary conditions. We will here present a method for deriving boundary conditions that are fully absorbing for waves with a certain angle of incidence at the boundary and very low reflecting for other angles.

This technique is applied to produce highly absorbing boundary conditions for a scalar and an elastic wave equation where time is the evolution direction. In this way one can use much smaller regions in the computations without having the experiment ruined by boundary reflections. Our technique is also applied to give absorbing side boundary conditions for the Claerbout migration equation. A proper treatment of the side boundaries is especially important when migrating before stacking and in 3-dimensional migration with few values in one coordinate direction. Together with the differential equations we also present difference formulas and the results from some numerical tests.

Before we consider the different differential equations, let us briefly sketch the underlying ideas. The absorbing boundary condition will be chosen such that its dispersion relation is a good approximation of the interior dispersion relation for outgoing waves. We must also be careful so that the boundary conditions together with the differential equation form a well posed problem. The better the boundary conditions describe outgoing waves the smaller will be the reflection. By approximating the dispersion relation with rational functions, we have a systematic way of deriving low reflecting conditions.

Another way of expressing this approach is to say that one way wave equations are used as boundary conditions. When time is the evolution direction the differential equation is used to predict the interior for later times. The boundary equation then predicts the values at the boundaries from the interior by describing outgoing waves.

Absorbing and reflecting boundary conditions for general differential equations and singular waves are rigorously analyzed in some recent papers (Nirenberg; Majda, Osher). Their boundary conditions are non-local and given by pseudo differential operators. Non local conditions cannot, however, be used in practical calculations. Local approximation of their operators essentially amounts to the approach we outlined above.

Let us now consider the scalar wave equation (1) where we assume v to be constant

$$\frac{1}{v^2} P_{tt} = P_{xx} + P_{zz}$$

$$t \geq 0 \ , \ 0 \leq x \leq 1 \ , \ z \geq 0 \tag{1}$$

Initially (at $t=0$) P and P_t are given. We will look for low reflecting conditions at $x=0$. (The boundary $x=1$ can be treated analogously.) The dispersion relation for (1) is

$$\frac{1}{v^2} \omega^2 = k_x^{\,2} + k_z^{\,2} \tag{2}$$

For waves traveling in the negative x direction, that is outgoing waves, the dispersion relation is

$$k_x = -\frac{\omega}{v}\left(1 - \left(\frac{v\,k_x}{\omega}\right)^2\right)^{1/2} \tag{3}$$

($-\omega$ is the dual of t)

The condition (3) in the frequency domain corresponds to a non local condition in the physical domain. In order to derive local boundary conditions we approximate (3) by rational functions with increasing order of accuracy

$$k_x = 0 \qquad (0^{th}\ \text{order}) \tag{4}$$

$$k_x + \frac{\omega}{v} = 0 \qquad (1^{st}\ \text{order}) \tag{5}$$

$$k_x + \frac{\omega}{v} - \frac{v^2 k_z^{\,2}}{2\omega} = 0 \qquad (2^{nd}\ \text{order}) \tag{6}$$

$$k_x + \frac{\omega}{v} \frac{1 - \frac{3}{4}\frac{v^2 k_z^2}{\omega^2}}{1 - \frac{1}{4}\frac{v^2 k_z^2}{\omega^2}} = 0 \qquad (3^{rd} \text{ order}) \qquad (7)$$

Compare SEP-8, p. 48 ff where the background for higher order approximations is given. These relations correspond to the following boundary conditions at $x = 0$.

$$L_0 P = P_x = 0 \qquad (4')$$

$$L_1 P = P_x - \frac{1}{v} P_t = 0 \qquad (5')$$

$$L_2 P = P_{tx} - \frac{1}{v} P_{tt} + \frac{v}{2} P_{zz} = 0 \qquad (6')$$

$$L_3 P = P_{ttx} - \frac{v^2}{4} P_{zzx} - \frac{1}{v} P_{ttt} + \frac{3v}{4} P_{tzz} = 0 \qquad (7')$$

The zero slope condition (4') is often called the reflective condition or the Neumann condition. In (5') a characteristic quantity is set to zero, and the equations (6') and (7') correspond to the 15° and 45° approximations respectively (Claerbout).

All of these relations come from approximations around horizontal incidence. We can also expand around some other angle analogous to the one way wave equation for slanted waves (see eg. SEP-8, p. 17 ff).

Let us check how good these boundary conditions are for traveling plane waves.

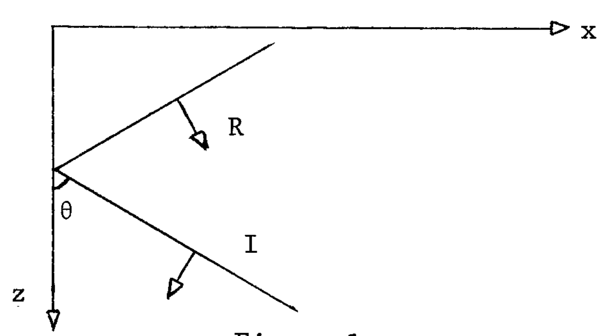


Figure 1.

From the incident wave

$$I = \exp (i (\omega t - k_x x + k_z z)) \tag{8}$$

with $\omega^2 = v^2 (k_x^{\,2} + k_z^{\,2})$ and the angle of incidence $\theta = \tan^{-1}(k_z/k_x)$ we get the reflected wave

$$R = A(\theta) \exp (i (\omega t + k_x x + k_z z)) \tag{9}$$

The waves (8) and (9) obey the differential equation (1). When the different boundary conditions are applied to the sum (P) of the waves we can determine $A(\theta)$.

$$L_0 (\exp (i (\omega t - k_x x + k_z z)) + \\ + A_0(\theta) \exp (i (\omega t + k_x x + k_z z))) = 0 \tag{10}$$

$$- k_x \exp (i (\omega t - k_x x + k_z z)) + \\ + A_0(\theta) k_x \exp (i (\omega t + k_x x + k_z z)) = 0$$

$$A_0(\theta) = 1 \quad \text{for} \quad x = 0 \tag{11}$$

The equations $L_i P = 0$ ($i = 1, 2, 3$) determine the other reflection coefficients in the same way (c stands for $\cos\theta$).

$$A_1(\theta) = \frac{c-1}{c+1}$$

$$A_2(\theta) = - \frac{1 - 2c + c^2}{1 + 2c + c^2} \tag{12}$$

$$A_3(\theta) = - \frac{1 - 3c + 3c^2 - c^3}{1 + 3c + 3c^2 + c^3}$$

$$(A_3 = A_1^3 \ , \ A_2 = - A_1^2)$$

In Figure 2 the absolute values of A_i ($i = 0, 1, 2, 3$) are displayed

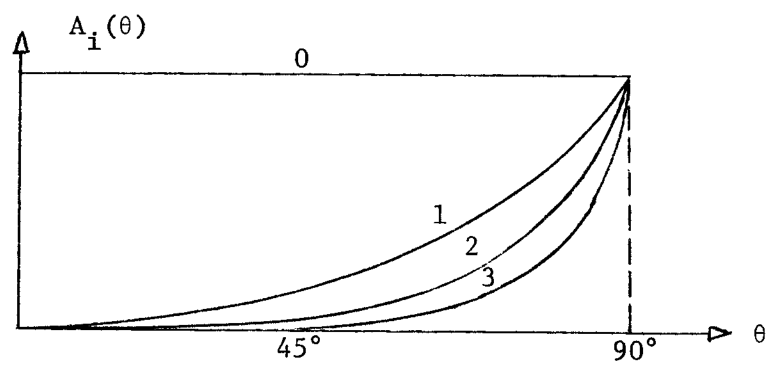


Figure 2.

As an example we have for $\theta = 45°$,

A_0	A_1	A_2	A_3
1.000	-0.172	- 0.029	- 0.005

Similarly we get $A_{-1}(\theta) = -1$ for the Dirichlet condition $P = 0$ at $x = 0$.

The figure displays the worst reflection for rays closest to glancing ($\theta \sim 90°$) . This is not too bad since those rays travel very slowly in the x direction. The strong reflection for large θ is also inherent in the problem. For any boundary operator L we have for $x = 0$

$$L\,(\exp(i(\omega t - k_x x + k_z z)) + A(\theta)\exp(i(\omega t + k_x x + k_z z))) = 0$$

$$A(\theta) = -\frac{c_1 - c_2 k_x + O(k_x^2)}{c_1 + c_2 k_x + O(k_x^2)}$$

In limes when $k_x \to 0$ ($\theta \to 90°$) we get $A = 1$ or $A = -1$ depending on whether $c_1 = 0$ or not. (c_1 and c_2 are independent of k_x . If $c_1 = c_2 = 0$ we can go further in the expansion and still derive $|A| = 1$.)

The equation

$$L_2' P = P_{tx} - \frac{1}{v} P_{tt} + v P_{zz} = 0$$

is a much better approximation than $L_2 P = 0$ (6') for waves with small k_x . As an absorbing boundary condition it is, however, worse. In fact, $A_2' = A_1$. This follows from the application of L_2' also to the reflected wave in (10).

The simplest normal mode test for well posedness is to be certain that no function

$$P = \exp(at + bx + ik_z z) \tag{13}$$

fulfills both the differential equation and the boundary condition for any a , b and k_z with $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) < 0$ and $\mathrm{Re}(k_z) = 0$. (Re stands for "real part of".) If there exists such a set of constants so that P is a solution to the initial boundary value problem, then also $\exp(\alpha(at + bx + ik_z z))$ is a solution for all $\alpha > 0$. Hence, we have solutions with arbitrary growth rate. The condition $\mathrm{Re}(b) < 0$ is for the initial data to be bounded ($t = 0$, $x \geq 0$) . This test gives a necessary condition which is valid for (4')-(7'). Let us here check on the second order approximation (for $v = 1$)

$$(1): \quad a^2 = b^2 - k_z^2 \tag{14}$$

$$(6'): \quad a\,b - a^2 - \frac{1}{2} k_z^2 = 0 \tag{15}$$

Insert k_z^2 from (14) into (15)

$$a\,b - a^2 - \frac{1}{2}(b^2 - a^2) = 0$$

$$-\frac{1}{2}(a-b)^2 = 0$$

We cannot have $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) < 0$ at the same time.

The discretizations we will suggest for the boundary conditions (4') -(7') are straightforward. They can be used together with any difference approximation of (1) . We let $P^n_{j,k}$ approximate $P(t_j, x_k, z^n)$ on a grid and describe the formulas with our standard difference operators ($t_j = j\,\Delta t$, $x_k = k\,\Delta x - \Delta x/2$, $z^n = n\,\Delta z$) .

$$D^x_+ P^n_{j,0} = 0 \tag{4''}$$

$$\left(D^x_+ - \frac{1}{v} D^t_- \right) P^n_{j,0} = 0 \tag{5''a}$$

$$D^x_+ (P^n_{j,0} + P^n_{j-1,0}) - \frac{1}{v} D^t_- (P^n_{j,0} + P^n_{j,1}) = 0 \tag{5''b}$$

$$D^x_+ D^t_0 P^n_{j,0} - \frac{1}{2v} D^t_+ D^t_- (P^n_{j,0} + P^n_{j,1}) + + \frac{v}{4} D^z_+ D^z_- (P^n_{j+1,1} + P^n_{j-1,0}) = 0 \tag{6''}$$

$$D^t_+ D^t_0 D^x_+ P^n_{j,k} - \frac{v^2}{8} D^z_+ D^z_- D^x_+ (P^n_{j-1,k} + P^n_{j+2,k}) - - \frac{1}{2v}(D^t_+)^2 D^t_- (P^n_{j,k} + P^n_{j,k+1}) + \frac{v}{8\Delta t} D^z_+ D^z_- (P^n_{j+2,k} + P^n_{j+2,k+1} - - P^n_{j-1,k} - P^n_{j-1,k+1}) = 0 \tag{7''}$$

All formulas but (5"a) are of second order. When the interior scheme is explicit the coupled scheme including the boundary condition will also be explicit if (4") to (6") are used. Stability analysis for the full initial boundary value problem is complicated. The absorbing nature of the boundary conditions makes it, however, easier to construct stable formulas.

The Figures 4-8 were all produced using the simple explicit difference approximation (16) in the interior

$$[\, D_+^t D_-^t - v^2 (\, D_+^x D_-^x + D_+^z D_-^z \,)\,]\, P_{j,k}^n \;=\; 0 \qquad (16)$$

We had roughly 10 points per wave in both x and z. The relation between the stepsizes were $4\,\Delta t = \Delta x = \Delta z$ for $v = 1$.

In Figures 4-6 we used the reflecting condition (4"). Figure 4 displays the initial waveform which was the start for all experiments. In Figure 5 the wave has expanded through 40 time steps and Figure 6 is after 100 time steps. Both Figures 7 and 8 show the effect of good absorption at the boundary after 100 time steps. In 7 the condition (5"a) was used and in 8 the condition (6"). There is a weak negative reflection in 7 which cannot be seen in 8.

The methods which we have developed for the scalar wave equation also carry over to the <u>elastic equation</u>.

$$\begin{aligned} \rho\, u_{tt} &= (\lambda + 2\mu)\, u_{xx} + (\lambda+\mu)\, w_{xz} + \mu\, u_{zz} \\ \rho\, w_{tt} &= \mu\, w_{xx} + (\lambda+\mu)\, u_{xz} + (\lambda + 2\mu)\, w_{zz} \end{aligned} \qquad (17)$$

$$t \geq 0 \;,\quad 0 \leq x \leq 1 \;,\quad z \geq 0$$

The analysis we need is developed in this report (p. 125 ff) regarding dispersion relations and one way wave equations and (p. 141 ff) for difference approximations. With the notation from these papers the first order and second order absorbing boundary conditions at $x = 0$ are

$$U_x - \begin{pmatrix} A_{22} & 0 \\ 0 & A_{11} \end{pmatrix} U_t = 0 \tag{18}$$

$$U_{tx} - \begin{pmatrix} A_{22} & 0 \\ 0 & A_{11} \end{pmatrix} U_{tt} + \begin{pmatrix} 0 & B_{21} \\ B_{12} & 0 \end{pmatrix} U_{tz} - \begin{pmatrix} C_{22} & 0 \\ 0 & C_{11} \end{pmatrix} U_{zz} = 0 \tag{19}$$

Compare formula (10) (SEP-10, p. 129) . We can directly use the analysis in the paper on one way elastic waves if we change the order of u and w in the equation (2), p. 125 .

$$\begin{pmatrix} w \\ u \end{pmatrix}_{tt} = D_1 \begin{pmatrix} w \\ u \end{pmatrix}_{zz} + H \begin{pmatrix} w \\ u \end{pmatrix}_{xz} + D_2 \begin{pmatrix} w \\ u \end{pmatrix}_{xx}$$

If x is now regarded as the extrapolation direction for upcoming waves the formulas in this report, p. 127 can be applied directly to give (18) and (19). Both conditions are absolutely absorbing for traveling waves at normal incidence. We can derive similar conditions even if we start from other formulations of the elastic wave equation.

We suggest the following explicit difference approximations of (18) and (19)

$$D_+^x \left(U_{j,0}^n + U_{j+1,0}^n \right) - \bar{A}\, D_+^t \left(U_{j,0}^n + U_{j,1}^n \right) = 0 \tag{18'}$$

$$D_0^t D_+^x U_{j,0}^n - \frac{1}{2}\bar{A} D_+^t D_-^t (U_{j,0}^n + U_{j,1}^n) +$$

$$+ \frac{1}{2}\bar{B} D_+^t D_0^z (U_{j-1,0}^n + U_{j,1}^n) - \frac{1}{2}\bar{C} D_+^z D_-^z (U_{j-1,0}^n + U_{j+1,1}^n) = 0 \qquad (19')$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ are the matrices in formula (19). These boundary conditions can then be coupled to any type of difference scheme used in the interior. We can also have an implicit version of (19') corresponding to formula (16), this report, p. 151 .

The next problem to be tackled is the migration equation which we state in the forward form

$$P_{tz} - \frac{v}{2} P_{xx} = 0 \qquad (20)$$

$$t \geq 0 \ , \quad 0 \leq x \leq 1 \ , \quad z \geq 0$$

The initial data is given for $z = 0$ and we want absorbing conditions at the side boundary $x = 0$. The dispersion relation for (20) is

$$\omega k_z + \frac{v}{2} k_x^2 = 0 \qquad (21)$$

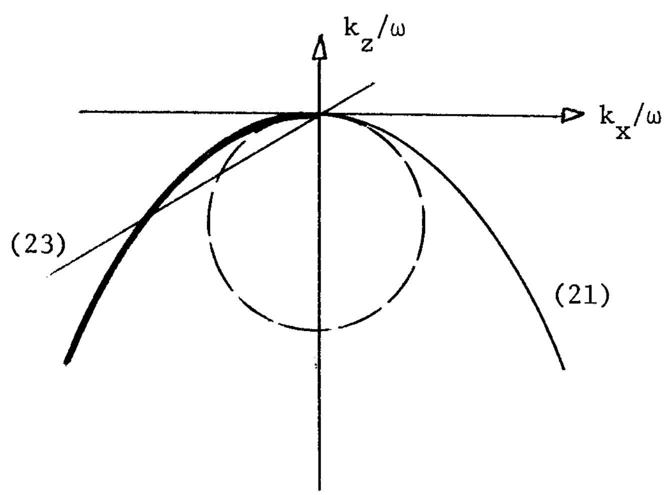


Figure 3.

We like to approximate the part of the parabola corresponding to outgoing waves (the thick curve in Figure 3). The circle in Figure 3 is the dispersion relation for the full scalar wave equation in retarded time coordinates, $t' = t - z/v$.

$$\frac{k_x}{\omega} = -\left(-\frac{2\, k_z}{v\, \omega}\right)^{1/2} \tag{22}$$

A simple approximation to the square root is

$$\left(-\frac{2\, k_z}{v\, \omega}\right)^{1/2} \sim -c\,\frac{k_z}{\omega}$$

The constant c is chosen to best handle those frequencies that appear in a particular calculation. The corresponding boundary condition will then be

$$P_x - c\, P_z = 0 \tag{23}$$

If we allow a constant term in the approximation of the square root we get a more general dispersion relation and boundary condition.

$$\frac{k_x}{\omega} + c_1 - c_2\,\frac{k_z}{\omega} = 0 \tag{24}$$

$$P_x - c_1\, P_t - c_2\, P_z = 0 \tag{25}$$

There is nothing that forces the approximations of (22) to be linear. In this report, p. 10 ff , Claerbout suggests a rational approximation in order to reduce the reflections.

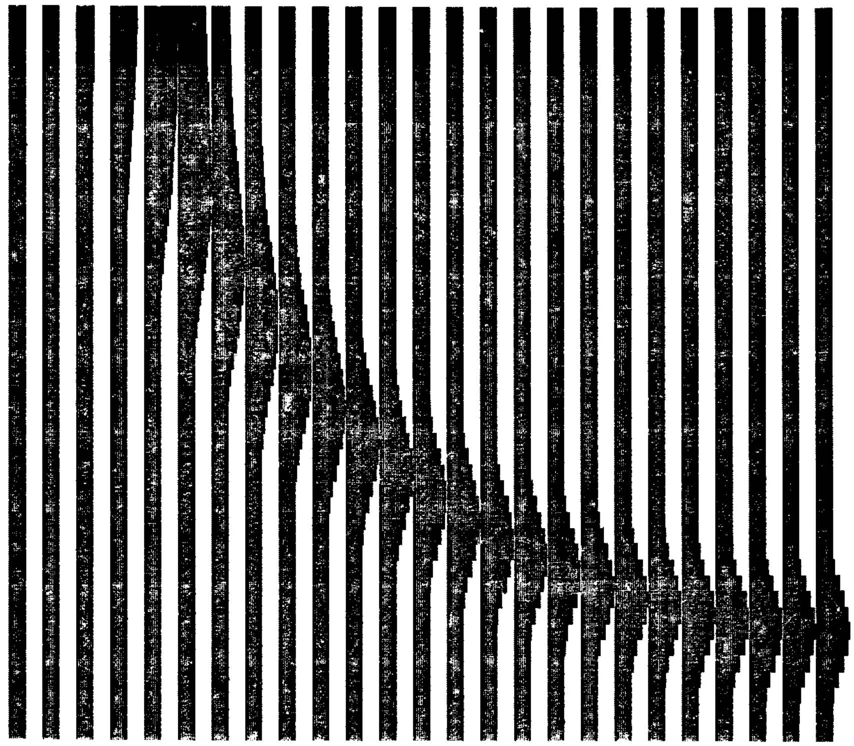
Figures 9-11 show migration of an impulse (given in Figure 9). The zero slope boundary condition ($P_x = 0$) was used at both side boundaries in Figure 10. At the left boundary in the last figure, we tested the absorbing condition (23) approximated by the difference formula

$$D_+^x P_{j,0}^n - c D_-^z P_{j,0}^n = 0 \qquad (26)$$

The constants c and v in the experiment were 1.5 and 1 respectively. We iterated for 100 z steps and used a standard explicit scheme in the interior.

We have so far in this paper derived boundary conditions by approximating the dispersion relation of differential equations. This is all right as long as we have a reasonable number of points per wavelength. Then the dispersion relation of the differential equation and the corresponding difference scheme are close to each other. If we have few points per wave we will be better off by adjusting the coefficients in the boundary conditions to fit the dispersion relation of the interior difference equation.

We tested these ideas on the one dimensional equation $P_{tt} = P_{xx}$ ($t \geq 0, x \geq 0$) with the analytically absolutely absorbing condition $P_x = P_t$ at $x = 0$. The reflection for high frequency waves could be reduced by a factor of 2-4 .



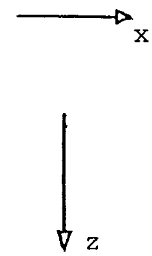


Figure 4. Scalar wave equation (1), initial values.

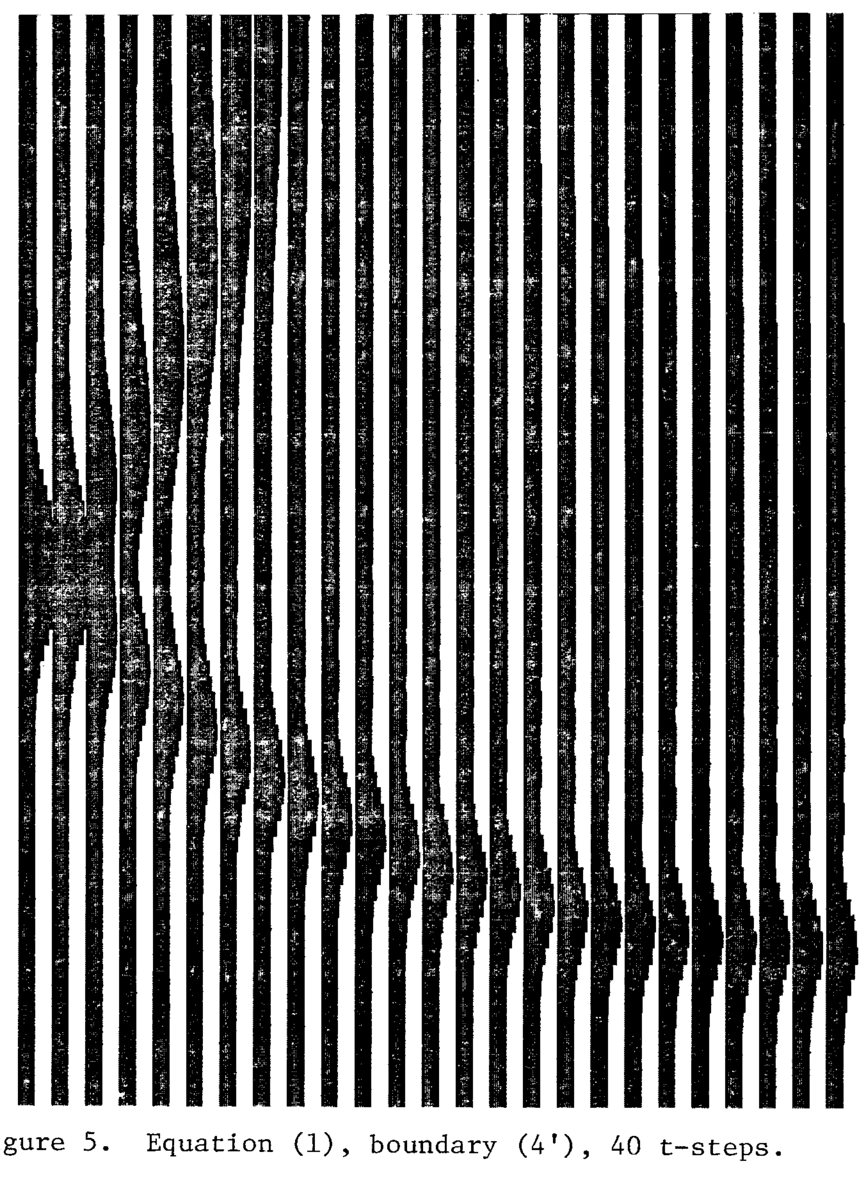


Figure 5. Equation (1), boundary (4'), 40 t-steps.

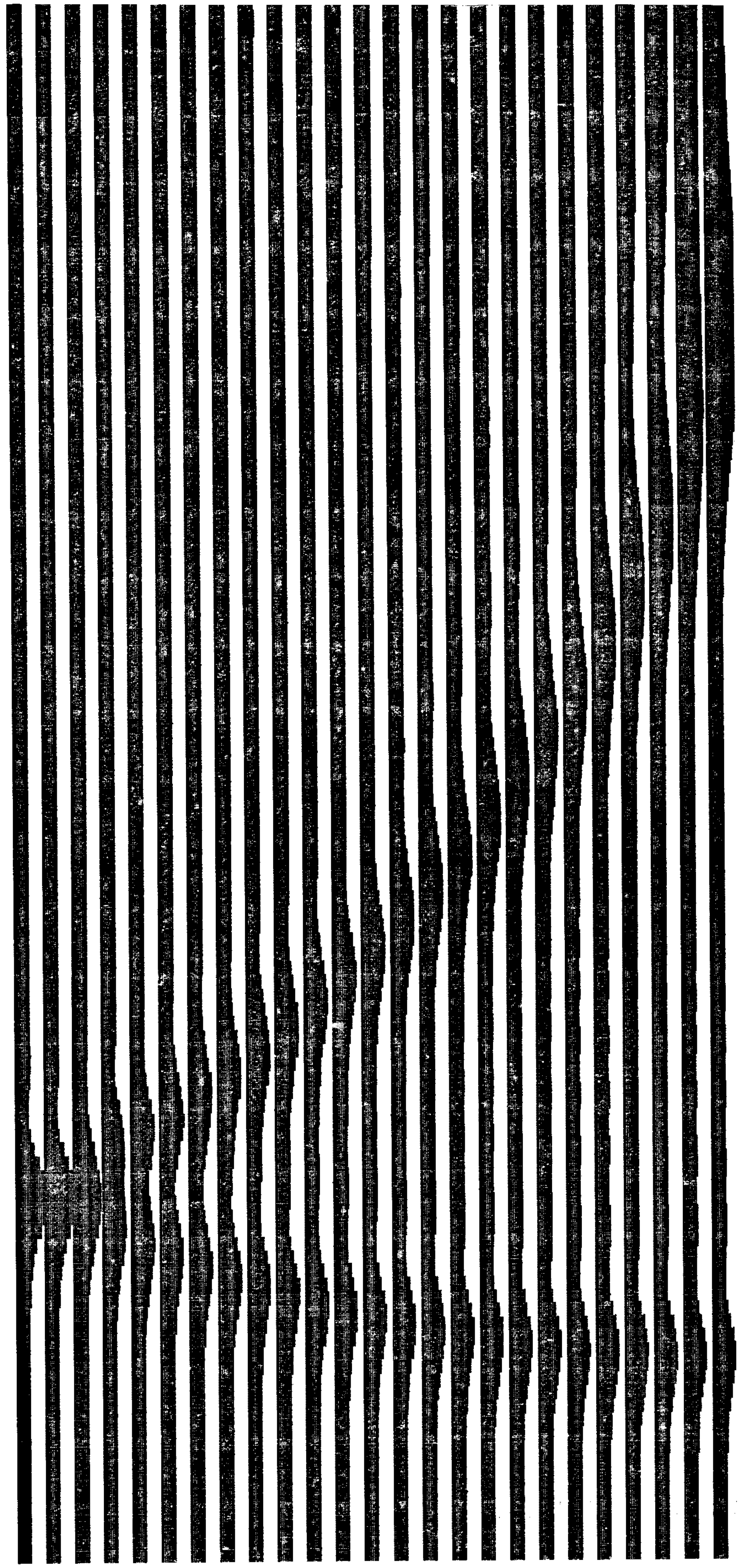


Figure 6. Equation (1), boundary (4'), 100 t-steps.

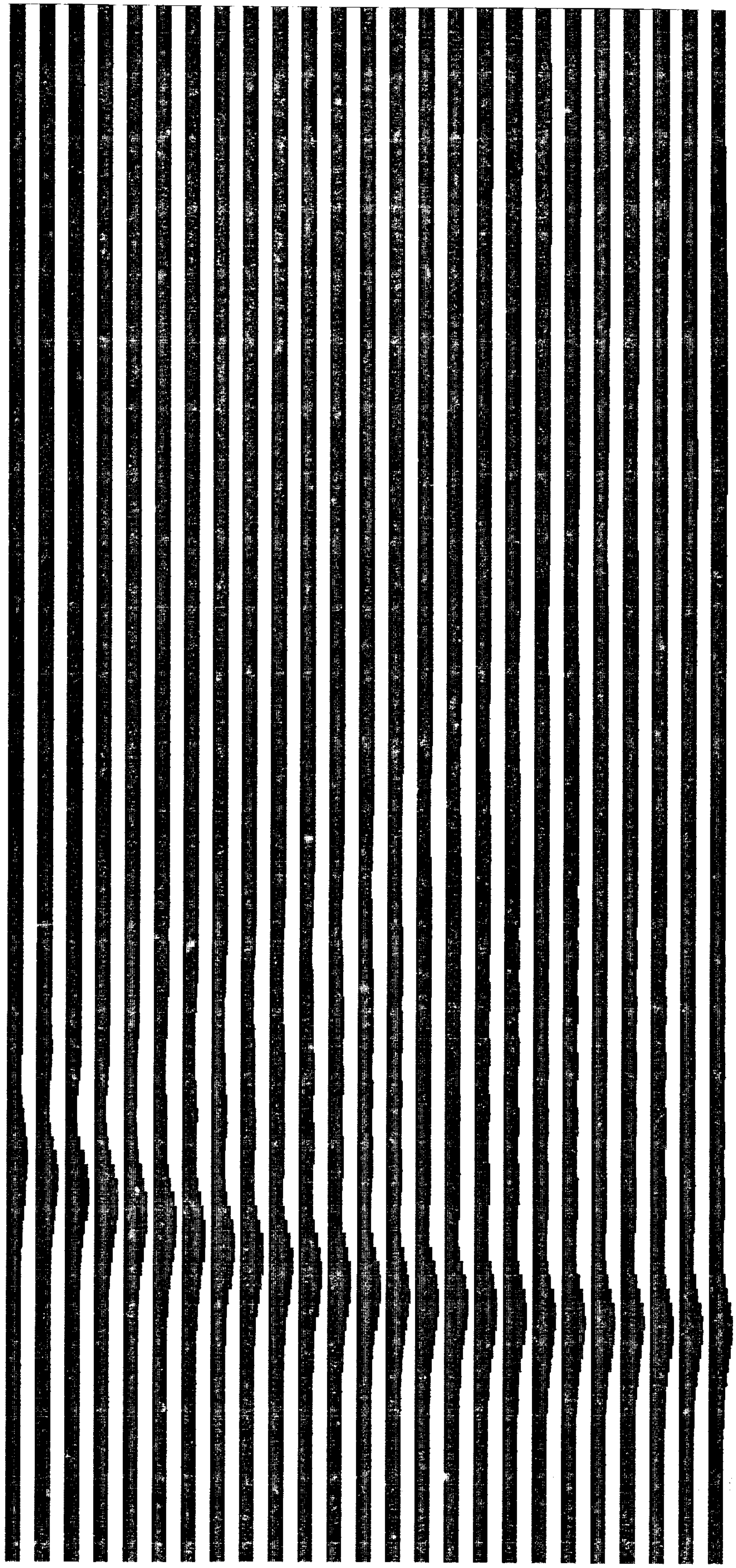


Figure 7. Equation (1), boundary (5'), 100 t-steps.

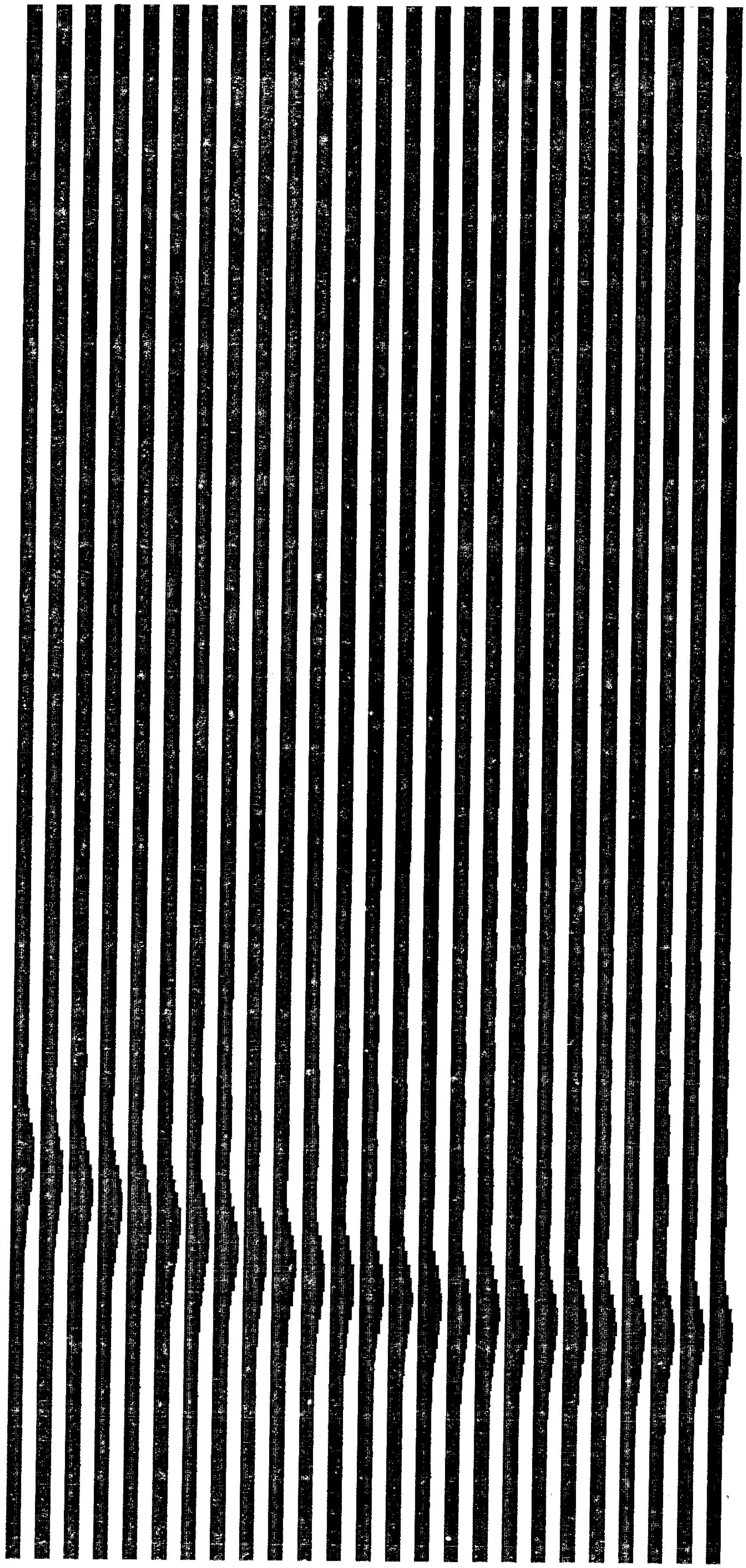


Figure 8. Equation (1), boundary (6'), 100 t-steps.

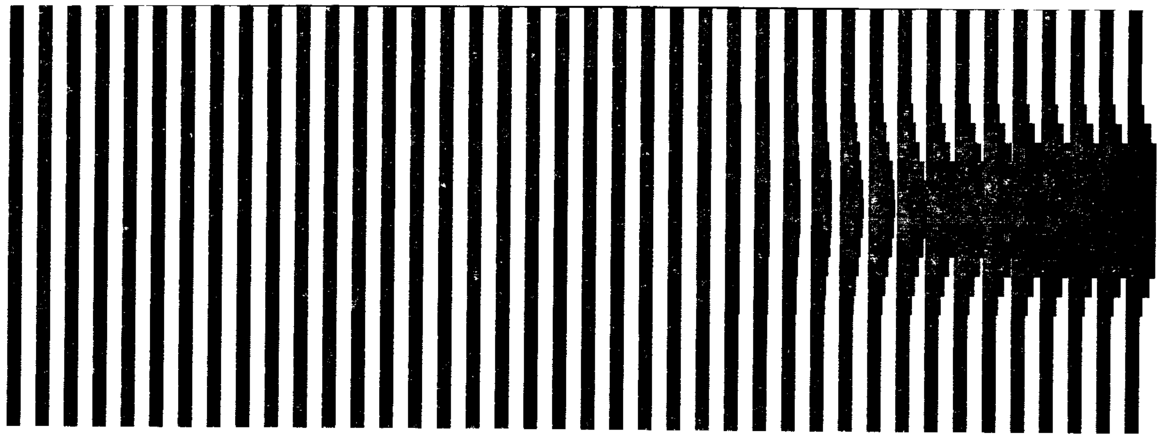


Figure 9. Migration, initial values, (x,t) plane

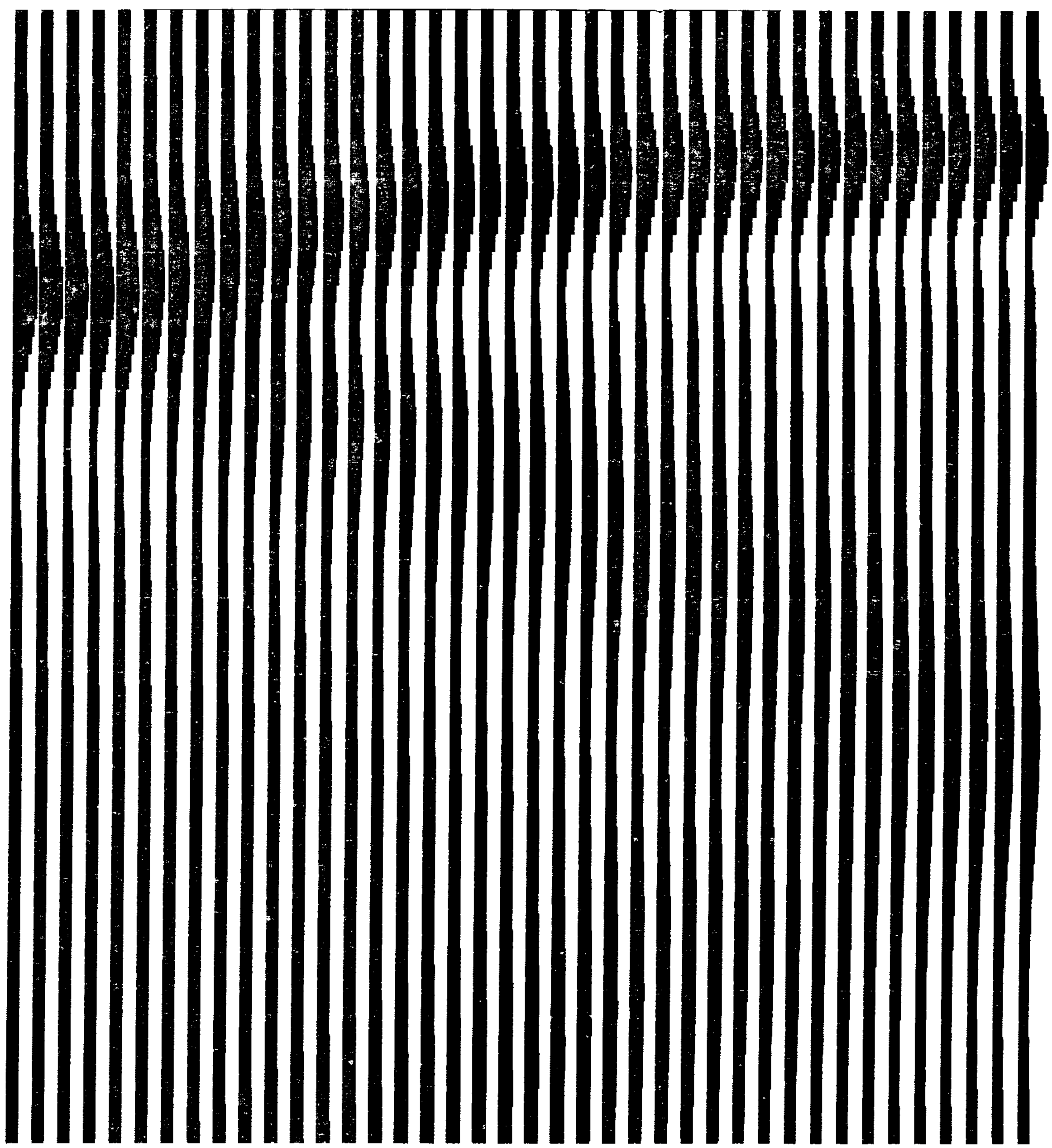


Figure 10. Migration, 100 z-steps, reflecting boundary.

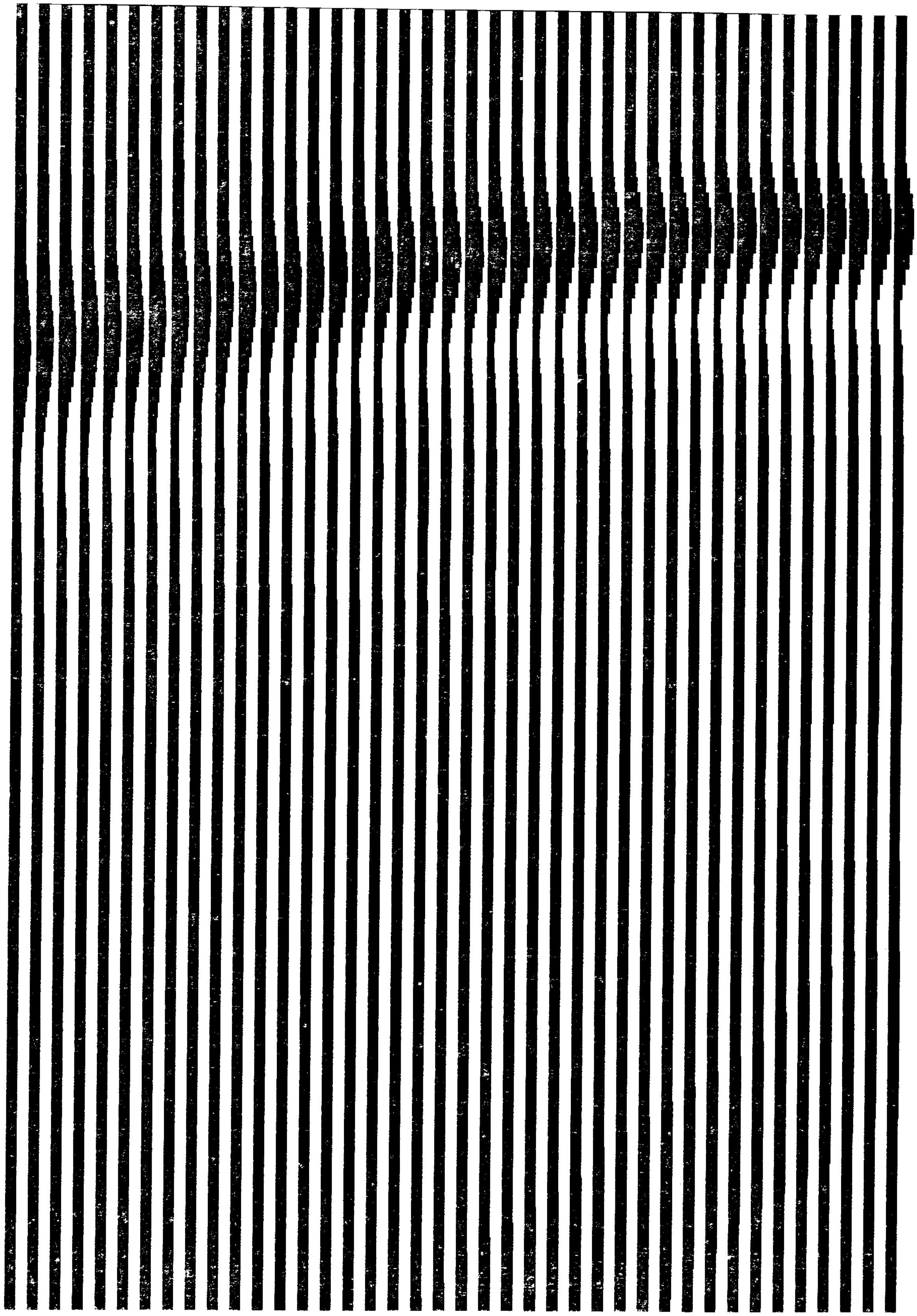


Figure 11. Migration, 100 z-steps, absorbing boundary.

References

[1] Nirenberg, L.: Lectures on Partial Differential Equations, Regional Conference Series #17, A.M.S., Providence, Rhode Island.

[2] Majda, A. and S. Osher: Reflection of Singularities at the Boundary, Comm. Pure Appl. Math., Vol. 28, 1975, pp. 479-499.

[3] Claerbout, Jon F.: Fundamentals of Geophysical Data Processing: With Application to Petroleum Prospecting, McGraw-Hill, Inc., 1976.