A Brief Derivation of the Dix Theorem

by Jon Claerbout

The equation for a circle in (x,z) space or a hyperbola in (x,t) space is

$$x^2 + z^2 = \bar{v}^2 t^2 \tag{1}$$

The question is this: Suppose waves propagate outwards from a surface point source in a stratified medium. The reflected waves are then fit, by some procedure, to a hyperbola and a $\bar{v}$ is determined. In what sense does $\bar{v}$ represent an average of the velocities in the layers? Begin by differentiating (1) with respect to x at a constant value of z.

Thus,

$$2x = \bar{v}^2 \, 2t \left(\frac{\partial t}{\partial x}\right)_z$$

or

$$\bar{v}^2 = \frac{x}{t} \left(\frac{\partial t}{\partial x}\right)_z^{-1} \tag{2}$$

From Figure 1 we see that in any layer the sine of the ray angle from the vertical is given by

$$\sin\theta = \frac{v\,dt}{dx}$$

which we can recognize as

$$\left(\frac{\partial t}{\partial x}\right)_z = \frac{\sin\theta\,(z)}{v(z)} = p \neq p(z) \tag{3}$$

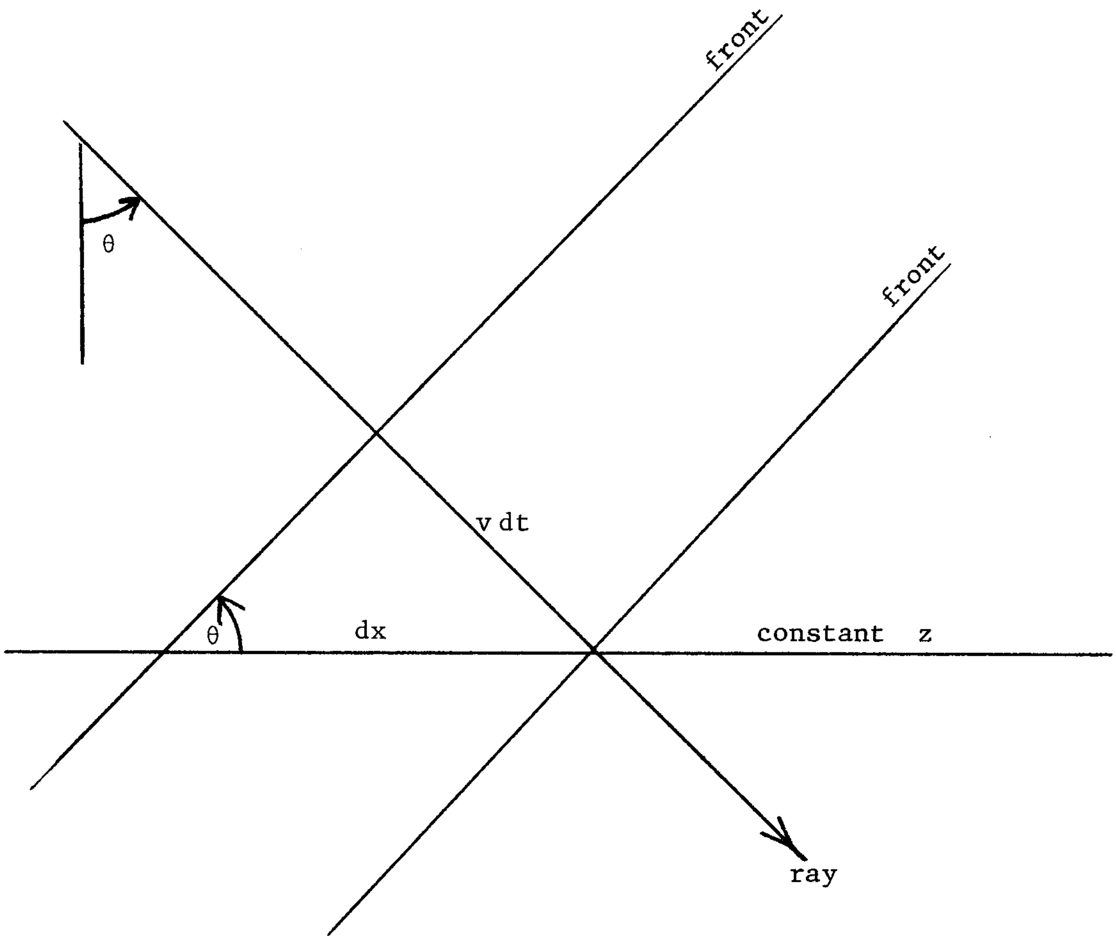

Figure 1. Diagram to illustrate that

$$p = \frac{\sin\theta}{v} = \left(\frac{\partial t}{\partial x}\right)_z$$

We wish to develop a mapping between the (x,z) coordinates in the vicinity of the source and (p,t) coordinates defined as ray parameter and travel times of rays leaving the origin $(x,z) = (0,0)$. Let the velocity $v(x,z) = v(z)$ be representable in either frame

$$v(x,z) = v(z) = v'(p,t)$$

Now the x coordinate of the tip of the ray as a function of (p,t) will be

$$x(p,t) = \int_0^t v'(p,t) \sin\theta(p,t) \, dt$$

$$= p \int_0^t v'(p,t)^2 \, dt \qquad (4)$$

Inserting (3) and (4) into (2) we have

$$\bar{v}^2 = \frac{1}{t} \int_0^t v'(p,t)^2 \, dt \qquad (5)$$

This shows that $\bar{v}$ is the root-mean-square velocity of the wave along its path. Notice that the "straight ray" approximation which occurs in some derivations is not really necessary. Of course the data will not be an exact hyperbola. But, if it is windowed about some particular x and t and $\partial t / \partial x$ is measured in that window then the $\bar{v}$ determined in that window will be exactly the RMS velocity for that particular ray (p).

From study of Figure 2, you should be able to recognize that

$$t'(p,d) \;=\; 2\,\tau(p,d) - 2\,p\,x(p,d) \tag{11}$$

Using (9) and (10) we find

$$t'(p,d) \;=\; 2\int_0^d \left(\frac{1}{v\,\cos\theta} - p\,\tan\theta \right) dz$$

$$t'(p,d) \;=\; 2p\int_0^d \left(\frac{1}{\sin\theta\,\cos\theta} - \frac{\sin^2\theta}{\sin\theta\,\cos\theta} \right) dz$$

$$t'(p,d) \;=\; 2p\int_0^d \frac{\cos\theta}{\sin\theta}\, dz \tag{12}$$

Inverse interpolation enables us to construct a table $d(p,t')$ from (12). The lateral shift of interest is

$$\Delta x(p,t') \;=\; \int_0^{d(p,t')} \tan[\theta(z)]\; dd \tag{13a}$$

$$= \int_0^{t'} \tan(\theta)\, \frac{dd}{dt'}\, dt' \tag{13b}$$

Differentiating (12) gives dt'/dd which reduces (13b) to Schultz's result

$$\Delta x(p,t') \;=\; \frac{1}{2p}\int_0^{t'} \tan^2[\,\theta(p,t')\,]\, dt' \tag{14}$$

Equation (14) is what you need when you have downward continued data in (x',t') space and you wish to assume a velocity so that you can laterally shift the data in order to display it in (x,t') space.