Expansion About Dipping Waves

by Jon F. Claerbout

We seek accurate rational approximations for the square root in the semicircle dispersion relation for upcoming waves

$$\frac{v\,k_z}{\omega} = -\left[1 - \left(\frac{v\,k}{\omega} \right)^2 \right]^{1/2} = -S \qquad (1)$$

Once we have an accurate rational approximation to equation (1) it is an easy matter to clear out the fractions and identify $(-i\omega, ik, ik_z)$ with $(\partial_t, \partial_x, \partial_z)$ and then we have an accurate differential equation.

Define

$$X = \frac{v\,k}{\omega} \qquad (2)$$

So (1) contains

$$S = (1 - X^2)^{1/2} \qquad (3)$$

We have previously used the square root approximations

$$S_1 = 1 \qquad (4a)$$

$$S_2 = 1 - \frac{1}{2} X^2 \qquad (4b)$$

$$S_3 = \frac{1 - \frac{3}{4} X^2}{1 - \frac{1}{4} X^2} \qquad (4c)$$

Francis Muir has pointed out that the square root approximations in (4) and generalizations to higher order may be generated by the recurrance

$$S_{j+1} = 1 - \frac{X^2}{1 + S_j} \qquad (5)$$

To check we presume convergence by setting S_{j+1} equal S_j in (5) getting

$$S(1+S) = 1 + S - X^2$$
$$S = (1 - X^2)^{1/2}$$

The approximations S_j may be developed by a recurrance technique for the numerators and denominators. Let

$$S_j = \frac{T_j}{B_j} \qquad (6)$$

Then (5) becomes

$$\frac{T_{j+1}}{B_{j+1}} = 1 + \frac{-X^2}{1 + \frac{T_j}{B_j}} = \frac{T_j + B_j - X^2 B_j}{T_j + B_j}$$

So we have the recurrance

$$T_{j+1} = T_j + (1 - X^2) B_j \qquad (7a)$$

$$B_{j+1} = T_j + B_j \qquad (7b)$$

As a check we have

j	T	B
1	1	1
2	$2 - X^2$	2
3	$4 - 3X^2$	$4 - X^2$

Curves of phase velocity and semicircle wavefront approximations are shown in Figure 1 . Figure 2 shows various hyperbola approximations in (x,t) space and in the retarded-time space (x',t') . The program which generated these figures, along with a generalization to slanted frames is included later. One of the obvious features of these plots is that the higher order approximations do an increasingly better job with the steeply dipping waves. The trouble with the higher order approximations is that they involve increasing amounts of computation. For that reason we will not consider a shift in the expansion point of the square root in order to squeeze a little more accuracy out of the low order approximations. Figure 3 shows a semicircle approximation which does not fit exactly at the point $(k, k_x) = (0, \omega/v)$. Instead, it fits exactly at the points $(k, k_z) = (\pm \sin\theta_0, \cos\theta_0)\, \omega/v$.

To develop such approximations we make some new definitions and we generalize our definition of X for nonzero θ .

$$s = \sin \theta_0 \tag{8a}$$

$$c = \cos \theta_0 \tag{8b}$$

$$X = \frac{k\,v}{c\,\omega} \tag{9a}$$

$$Y = \left(\frac{kv}{c\omega} \right)^2 - \frac{s^2}{c^2} = X^2 - \frac{s^2}{c^2} \tag{9b}$$

In terms of these definitions the semicircle dispersion relation (1) becomes

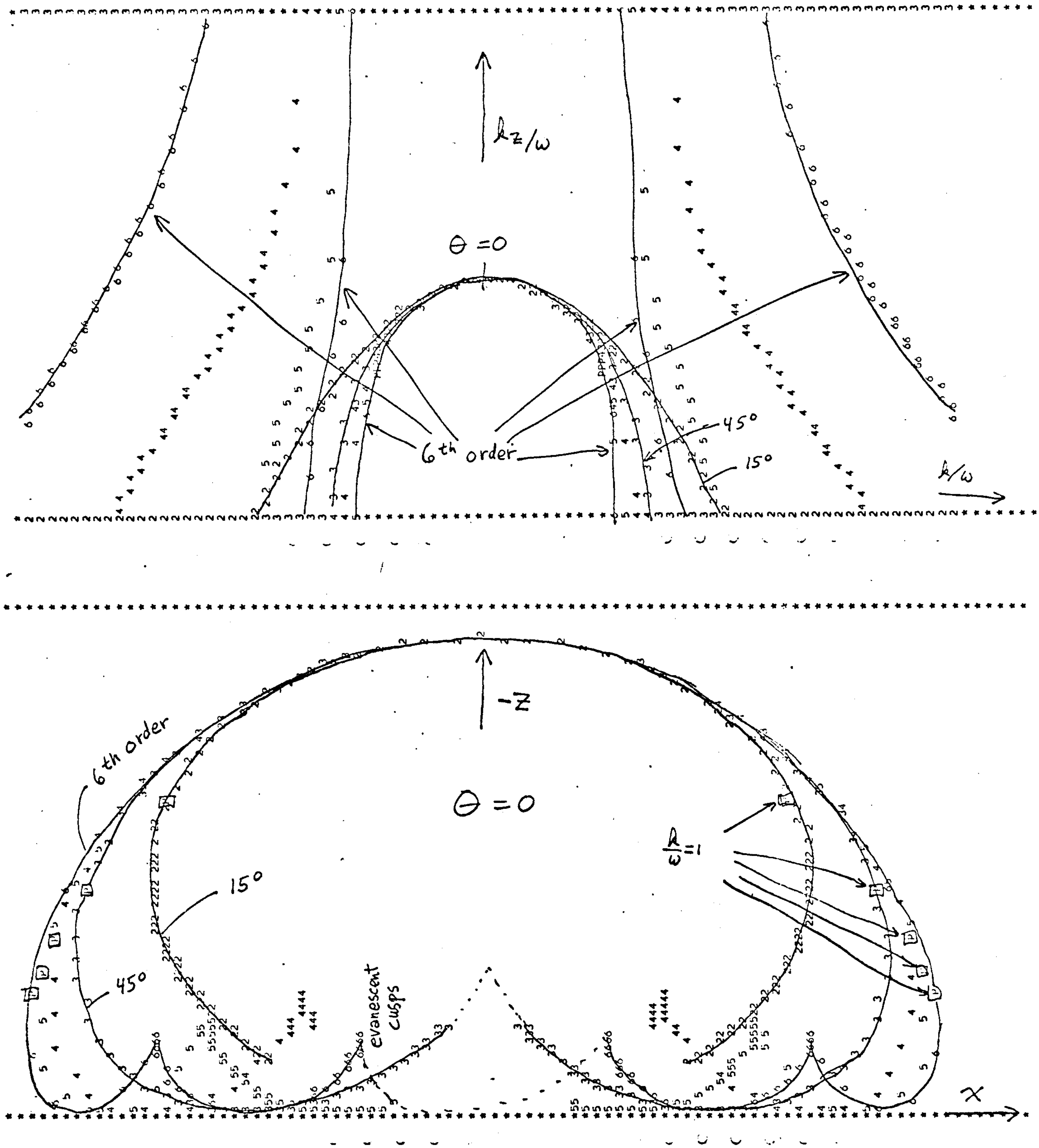

Figure 1. Phase velocity approximations to a semicircle (k_z/ω, k_x/ω), top. Group velocity approximations, equation (35c), bottom. All for tangency angle $\theta = 0$.

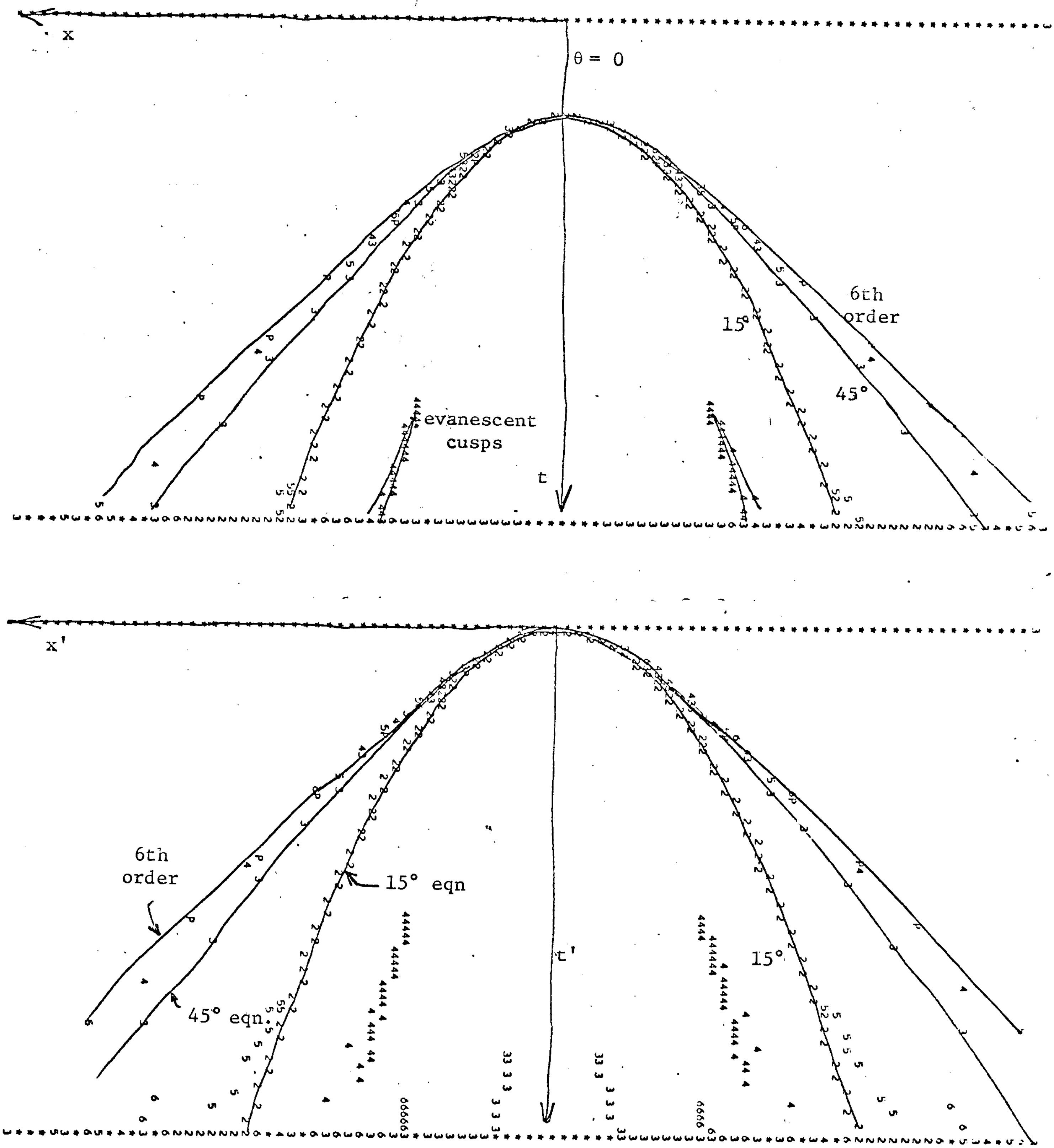

Figure 2. Approximations to a hyperbola. All fit exactly at the apex ($\theta=0$) . Top is the real (x,t) space (equation 35a), and bottom is the computational (x',t') space (equation 35b), in which the hyperbolas are shifted to be tangent to $t' = 0$.

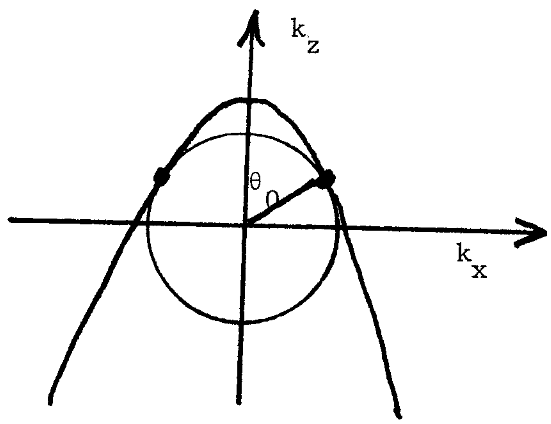

Figure 3. Dispersion relation for best fit to dipping waves.

$$\frac{v\,k_z}{\omega} = -(1-X^2)^{1/2}$$

$$= -[\,1-(\,c^2Y+s^2\,)\,]^{1/2}$$

$$= -[\,(1-s^2)-c^2\,Y\,]^{1/2}$$

$$= -c\,(1-Y)^{1/2}$$

or

$$\frac{v\,k_z}{c\,\omega} = -(1-Y)^{1/2} = -S \tag{10}$$

For S_1 we have

$$\frac{v\,k_z}{\omega\,c} = -1+\frac{Y}{2}$$

$$= -1-\frac{s^2}{2c^2}+\frac{1}{2}\left(\frac{kv}{c\omega}\right)^2$$

or

$$\omega\,k_z = -\left(c+\frac{s^2}{2c}\right)\frac{\omega^2}{v}+\frac{v}{2c}\,k^2 \tag{11}$$

We can quickly convert this to a partial differential equation in (t, x, z) space, but we would rather have one in the retarded time space (t', x', z'). The trick is to be able to define retarded time t' in such a way that the shifting term [the ω^2 term in (11)] drops out. To do this we define the transformation

$$t' = t + \frac{\alpha}{v} z \tag{12a}$$

$$x' = x \tag{12b}$$

$$z' = z \tag{12c}$$

As we have seen, the chain rule for differentiation provides us with

$$\omega = \omega' \tag{13a}$$

$$k = k' \tag{13b}$$

$$k_z = k'_z - \frac{\alpha}{v} \omega' \tag{13c}$$

Substitute (13) into (11)

$$\omega' (k'_z - \frac{\alpha}{v} \omega') = - (c + \frac{s^2}{2c}) \frac{\omega'^2}{v} + \frac{v}{2c} k'^2 \tag{14}$$

Recognize that a choice of $\alpha = c + s^2/(2c)$ eliminates the shift term, reducing (14) to

$$\omega' k'_z = \frac{v}{2c} k'^2$$

$$(-i\omega')(i k'_z) = - \frac{v}{2c} (ik)^2$$

which inverse transforms to the partial differential equation

$$P'_{z't'} = - \frac{v}{2 \cos \theta_0} P'_{x'x'} \tag{15}$$

We note that a practical prescription for broadening the range of angles adequately migrated is to over-migrate by a factor of $1 / \cos \theta_0$. The error at $k = 0$ is given by the departure of α from 1 . For $\theta_0 = 20°$ we find an acceptable $\alpha = 1.0019$ at an over-migration of 6% .

Rather than continue manual calculation it seems preferable to develop a general technique and a program to generate the various schemes along with their phase and group errors. Muir's recurrance for (3) obviously works for (10) as well. So we just grab (7) with X^2 replaced by Y .

$$T_{j+1} = T_j + (1 - Y) B_j \tag{14a}$$

$$B_{j+1} = T_j + B_j \tag{14b}$$

However, we don't really want T and B to come out to be polynomials in Y ; we want them as polynomials in X^2 . We could compute the polynomials by (14) and then shift the origin with (9b). However, we can also substitute (9b) into (14) to get the recurrance directly in terms of X^2 .

$$T_{j+1} = T_j + [(1 + \tan^2 \theta_0) - X^2] B_j$$

$$T_{j+1} = T_j + \left(\frac{1}{\cos^2 \theta_0} - X^2 \right) B_j \tag{15a}$$

$$B_{j+1} = T_j + B_j \tag{15b}$$

Now we have obtained the dispersion relation approximation to (10) in the form

$$\hat{k}_z = -\frac{\omega c}{v}\,\frac{T(X^2)}{B(X^2)} \tag{16}$$

To get into the retarded time frame we need to get rid of the coefficient of X^2 to the zero power in T. This is easily done as follows:

$$\hat{k}'_z = -\frac{\omega c}{v}\left(\frac{T(X^2)}{B(X^2)} - \frac{T_0}{B_0}\right) \tag{17}$$

where T_0 and B_0 are the coefficients of X^2 to the zero power in T and B. Clearly $c\,T_0/B_0$ is the generalized value of α. We now observe from (17) that

$$\frac{\partial\,\hat{k}'_z}{\partial\omega} = \frac{\partial\,\hat{k}_z}{\partial\omega'} - \frac{c}{v}\,\frac{T_0}{B_0} \tag{18}$$

This means that the hyperboloids in (x,t) space are shifted toward $t' = 0$ in (x',t') space.

The important thing about (18) is this: We have not yet proven that the high order schemes which we are developing will be stable. Earlier work proved that for $\theta = 0$ the 15° equation is stable. Experience showed that the 45° equation was stable at $\theta = 0$. We saw that for the 15° equation a choice of $\theta = 20°$ just amounted to a 6% over-migration, so there is obviously no instability there. It is my belief that the fact that (18) is everywhere of the same sign (negative), which can be seen on the computer outputs, ensures that the differential equation is stable so that stable difference schemes can be found. I believe this because of the theorem in time series analysis that the group delay of a causal all-pass filter is positive. This theorem is proven in my book on pages 39-42.