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Downward Continuation of Operators

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Introduction

There are various ways in which data can be downward continued, depending on whether the sources are moved along or not and whether the free surface is fixed or moved along. The first section will treat the one-dimensional problem, assuming a layered earth and plane wave propagation in the direction perpendicular to the layers. The second section will treat a more general case.

1. The One-Dimensional Case

The basic equation involved is

$$\begin{bmatrix} U_n \\ D_n \end{bmatrix} = L(n,0) \begin{bmatrix} U_0 \\ D_0 \end{bmatrix} \tag{1}$$

where

$$L(n,0) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

is defined by

$$L(n,0) = L(n,n-1)\, L(n-1,n-2) \ldots L(1,0) \tag{2}$$

$$L(k+1,k) = \frac{1}{\sqrt{z}(1+c_k)} \begin{bmatrix} 1 & c_k z \\ c_k & z \end{bmatrix}$$

We also assume that at some given depth m there are no more upgoing waves: $U_m \equiv 0$.

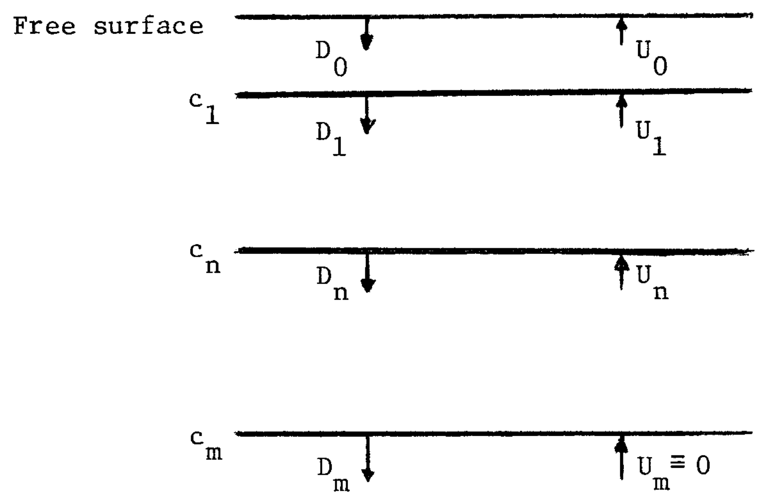

Figure 1. Layered earth model.

(i) Source and Free Surface Fixed

This is the usual downward continuation scheme (see Fig. 2).

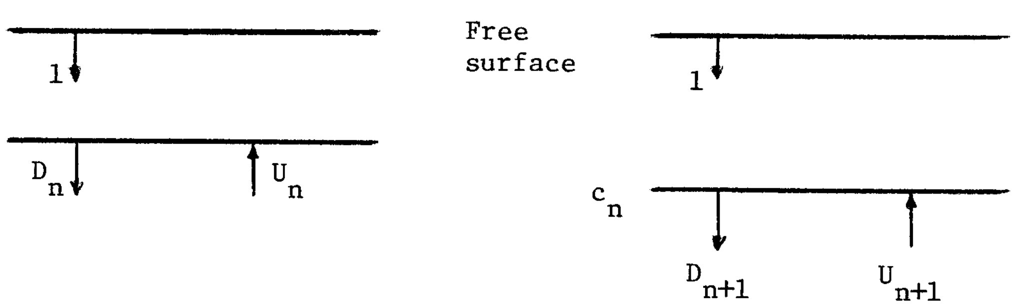

Figure 2. Source and free surface fixed.

The downward continuation operator is derived directly from equation (2):

$$U_{n+1}(z) = \frac{1}{\sqrt{z}(1+c_n)} [\ U_n(z) + c_n\ z\ D_n(z)\ ] \tag{3}$$

$$D_{n+1}(z) = \frac{1}{\sqrt{z}(1+c_n)} [\ c_n\ U_n(c) + z\ D_n(z)\ ]$$

(ii) Source and Free Surface Moving

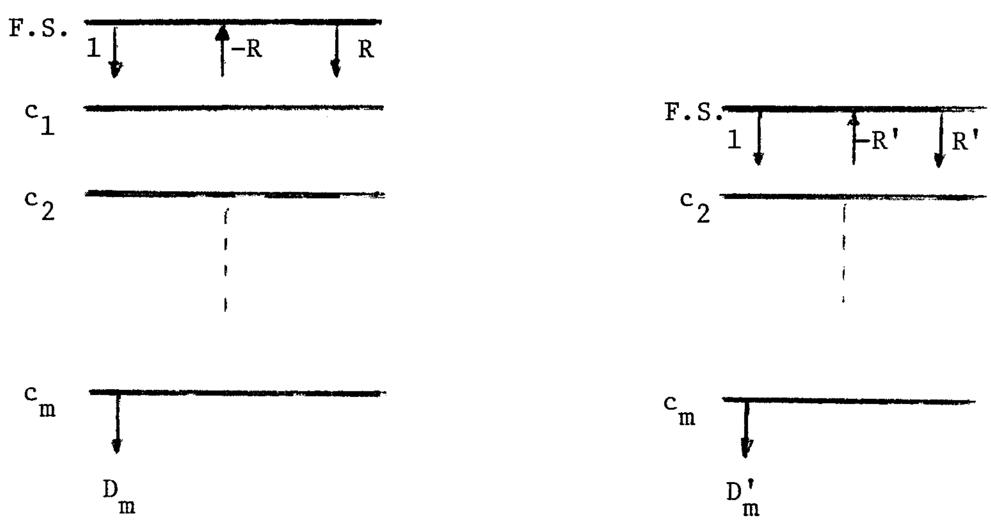

Figure 3. Free surface and source moving.

Using equations (1) and (2) with n=m we can write

$$\begin{bmatrix} 0 \\ D_m \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} -R' \\ 1+R' \end{bmatrix} \tag{4}$$

and

$$\begin{bmatrix} 0 \\ D'_m \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \frac{1}{\sqrt{z}(1+c_1)} \begin{bmatrix} 1 & c_1 z \\ c_1 & z \end{bmatrix} \begin{bmatrix} -R \\ 1+R \end{bmatrix} \tag{5}$$

which gives

$$\frac{L_{12}}{L_{11}} = \frac{R'}{1+R'} \tag{4'}$$

$$(L_{11} + c_1 L_{12})R = (L_{11}c_1 + L_{12}) z (1+R) \tag{5'}$$

which after some algebra gives

$$R'(z) = \frac{c_1 z + (1-c_1 z) R(z)}{(1+c_1)(z-1)R(z)+(1+c_1)z} \tag{6}$$

Multiplying through by the denominator, and comparing equal powers of z will give an algorithm to find $R'(z)$. The details are omitted.

(iii) Free Surface Fixed, Sources Moving

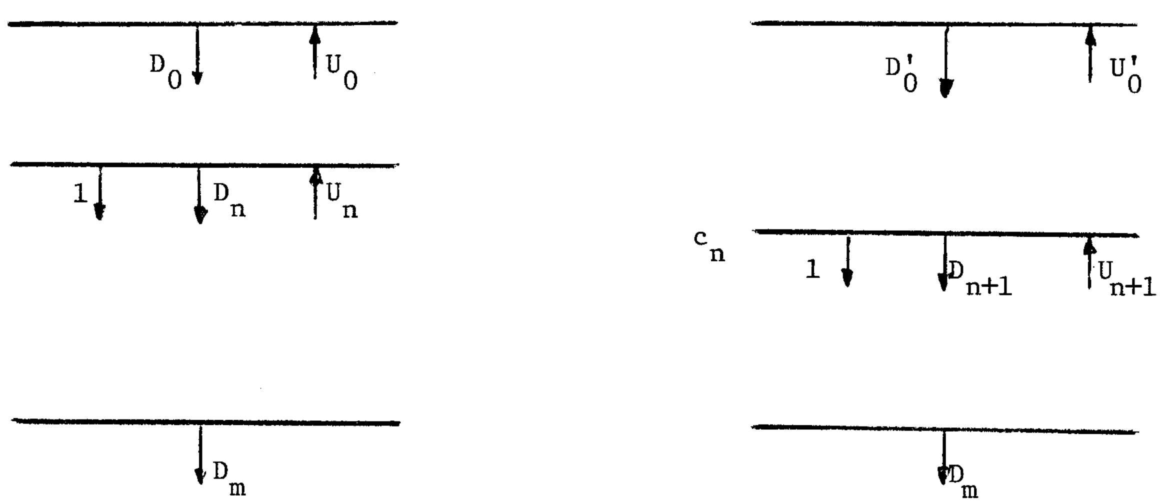

Figure 4. Free surface fixed, source moving.

Using equations (1), (2) we can write

$$\begin{bmatrix} U_n \\ D_n \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_0 \\ D_0 \end{bmatrix} \tag{7}$$

where $U_0 = - D_0$ and

$$\begin{bmatrix} U_{n+1} \\ D_{n+1} \end{bmatrix} = \frac{1}{\sqrt{z}(1+c_n)} \begin{bmatrix} 1 & c_n z \\ c_n & z \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_0' \\ D_0' \end{bmatrix} \tag{8}$$

where $U_0' = - D_0'$ or

$$\frac{U_n}{D_n} = \frac{L_{11} - L_{12}}{L_{21} - L_{22}} \triangleq Y_u \tag{7'}$$

$$\frac{U_{n+1}}{D_{n+1}} = \frac{Y_u + c_n z}{c_n Y_u + z} \triangleq Y_u' \tag{8'}$$

where Y_u , Y_u' are the admittances looking 'upwards'.

Similarly,

$$\begin{aligned} \begin{bmatrix} U_m \\ D_m \end{bmatrix} &= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} U_n \\ 1+D_n \end{bmatrix} = \\ &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \frac{1}{\sqrt{z}(1+c_n)} \begin{bmatrix} 1 & c_n z \\ c_n & z \end{bmatrix} \begin{bmatrix} U_n \\ 1+D_n \end{bmatrix} \end{aligned} \tag{9}$$

where $U_m = 0$ and

$$\begin{bmatrix} U'_m \\ D'_m \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} U_{n+1} \\ 1+D_{n+1} \end{bmatrix} \quad ; \quad U'_m = 0 \tag{10}$$

Defining

$$\begin{aligned} Y_d &= \frac{N_{12}}{N_{11}} = -\frac{U_n}{1+D_n} \\ Y'_d &= \frac{M_{12}}{M_{11}} = -\frac{U_{n+1}}{1+D_{n+1}} \end{aligned} \tag{11}$$

we get from (9)

$$Y_d = \frac{c_n z + Y'_d z}{1 + c_n Y'_d} \tag{12}$$

Combining (7'), (8'), (11), (12) we get

$$\begin{aligned} U_{n+1} &= \frac{1}{z U_n (1-c_n^2)} [(U_n + c_n z D_n)(U_n + c_n z D_n + c_n z)] \\ D_{n+1} &= \frac{1}{z U_n (1-c_n^2)} [(c_n U_n + z D_n)(U_n + c_n z D_n + c_n z)] \end{aligned} \tag{13}$$

Again, multiplying through by the denominator and comparing coefficients of powers of z gives a recursion for finding U_{n+1}, D_{n+1} from knowledge of U_n, D_n, c_n .

(iv) <u>Source Fixed, Free Surface Moved Up to Infinity</u>

This case is discussed in detail in Don Riley's thesis.

Figure 5. The 'Noah geometry'.

It is easy to show that

$$C(z) = \frac{R(z)}{1+R(z)} \tag{14}$$

and as before, we can find recursion to compute $C(z)$ from $R(z)$.

Some practical problems that arise with the type of recursion appearing in (ii), (iii), (iv) are mentioned in Don Riley's thesis, page 8-12.

2. Extension of the Results to a Two-Dimensional Model

Assume we are performing an experiment as in Figure 6.

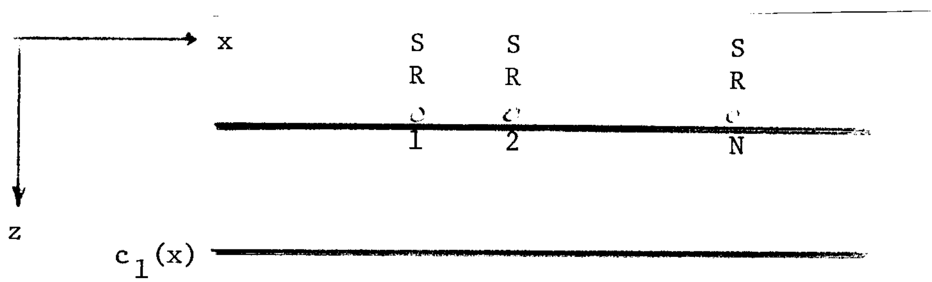

Figure 6. The two-dimensional case.

and we arrange the data from the different experiments in matrix form, i.e., $U_n(z)$, $D_n(z)$ are $N \times N$ polynomial matrices, where the (i,j) entry corresponds to source i and receiver j . Denote by M the delay and diffraction operator that downward (upward) continues $D_n(U_n)$. If we assume that interaction between the up and downgoing waves occurs only at the layer boundaries, we can get a similar formula to the one-dimensional case:

$$\begin{bmatrix} U_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} (I+C_1)^{-1}M^{-1} & (I+C_1)^{-1}M\,C_1 \\ (I+C_1)^{-1}C_1M^{-1} & (I+C_1)^{-1}\,M \end{bmatrix} \begin{bmatrix} U_0 \\ D_0 \end{bmatrix} \tag{15}$$

where C_1 is a diagonal matrix with $c_1(x)$ on the diagonal.

For a more explicit definition of M look at the differential equation

$$D_z = \frac{i\omega}{v} D + \frac{iv}{2\omega} D_{xx} \qquad (16)$$

Its solution will have the form

$$D(\Delta z) = e^{(\frac{i\omega}{v}+\frac{iv}{2\omega}\partial_{xx})\Delta z} D_0 = e^{\frac{i\omega}{v}\Delta z} e^{\frac{iv}{2\omega}\Delta z\,\partial_{xx}} D_0 \qquad (17)$$

Now:

$$e^{\frac{i\omega}{v}\Delta z} = e^{i\omega\Delta t} \cong z^{1/2} \qquad (18)$$

and

$$e^{\frac{iv}{2\omega}\Delta z\,\partial_{xx}} \cong \frac{1 + \frac{iv}{4\omega}\Delta z\,\partial_{xx}}{1 - \frac{iv}{4\omega}\Delta z\,\partial_{xx}} = \frac{I + \frac{iv}{4\omega}\frac{\Delta z}{(\Delta x)^2} T}{I - \frac{iv}{4\omega}\frac{\Delta z}{(\Delta x)^2} T} \qquad (19)$$

where

$$T = \begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & & \\ & & & 1 \\ 0 & & 1 & -2 \end{bmatrix}$$

Finally, using the bilinear transform, we get

$$M = z^{1/2} \frac{I + \frac{1+z}{1-z} a T}{I - \frac{1+z}{1-z} a T} \quad ; \quad a = \frac{1}{4} \left(\frac{\Delta z}{\Delta x} \right)^2$$

Using the facts developed in this section we can repeat the calculations of the previous section, following the same derivation.

If we make the assumption that C_1 , M and R commute, we get exactly the same results as in the scalar case, with $\sqrt{z}$ being replaced by M . For example, equation (6) becomes

$$[\ (I+C_1)(M^2-1)\ R(z) + (I+C_1)\ M^2\ ]\ R'(z) = \\ = C_1M^2 + (\ I + C_1\ M^2\ )\ R(z) \qquad (21)$$

By inserting M from equation (20), multiplying through by the denominator and comparing powers of z , we can get recursions as before. The other equations go through in a similar manner, and the details are omitted.

If we relax the assumption that the various matrices commute, the equations become more complicated.