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A Stratified Media Slant Frame

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When convolving hyperbolas on to CDP gathers to determine subsurface velocity, much time (both human and computer) is saved by using velocity gates in the analysis to exclude impossible or unlikely velocities. This is a case where a good idea of the range of the solution can be used as an input constraint to economize the calculation. In a similar manner we would like to obtain a set of transformation equations which will involve a vertically varying migration velocity, $\bar{v} = \bar{v}(z)$, which closely follows the vertical variation of the true velocity, $\hat{v}(x,z)$. This has the advantage that any further estimation refinements will be concerned only with deviations from $\bar{v}(z)$. An additional advantage is in the coupling of up and downgoing waves, which in slant frames, involves an implicit drifting in the x-direction. The rate of drift as a function of depth, z , is directly related to the local z-dependent velocity through the wave parameter, p .

$$p = \frac{\sin\theta(z)}{v(z)} = \text{constant} \tag{1}$$

Thus, our total migration picture, which necessarily involves U and D coupling, will be able to predict more exactly the position in (x,z) space where the waves interact.

We expect that stratified media frames involving an approximate migration velocity, $\bar{v}(z)$, will make both lateral velocity variation estimates and deep sea floor statics corrections more computationally efficient.

Estevez (SEP 5, p. 20-42) has discussed stratified media slant frames where the migration velocity, $\bar{v}(z)$, specifies the subsurface velocity structure exactly, i.e., $\tilde{v} = \bar{v}$.

Let us begin with the downgoing wave transformations for $\theta = 0$.

$$x' = x \tag{2a}$$

$$z' = z \tag{2b}$$

$$t' = t - \int_0^z \frac{dz}{\bar{v}(z)} \tag{2c}$$

where equation (2c) is an obvious generalization of $t' = t - z/\bar{v}$.

Now,

$$P_x = P'_{x'}$$

$$P_z = P'_{z'} - \frac{1}{\bar{v}(z)} P'_{t'}$$

$$P_t = P'_{t'}$$

and,

$$P_{xx} = P'_{x'x'}$$

$$P_{zz} = P'_{z'z'} - \frac{2}{\bar{v}(z)} P'_{z't'} + \frac{1}{\bar{v}^2(z)} P'_{t't'} + \frac{\bar{v}_z(z)}{\bar{v}^2(z)} P'_{t'}$$

$$P_{tt} = P'_{t't'}$$

Substitute into the wave equation

$$P_{xx} + P_{yy} + P_{zz} = \frac{1}{\tilde{v}^2(x,z)} P_{tt} \tag{3}$$

to obtain

$$P'_{x'x'} + P'_{z'z'} - \frac{2}{\bar{v}(z)} P'_{z't'} + \frac{1}{\bar{v}^2(z)} P'_{t't'} + \frac{\bar{v}_z(z)}{\bar{v}^2(z)} P'_{t'}$$

$$= \frac{1}{\tilde{v}^2(x,z)} P'_{t't'}$$

or,

$$P'_{z't'} = +\frac{\bar{v}(z)}{2} P'_{x'x'} + \frac{\bar{v}(z)}{2} \left(\frac{1}{\bar{v}^2(z)} - \frac{1}{\tilde{v}^2(x,z)} \right) P'_{t't'} + \frac{\bar{v}_z(z)}{2\bar{v}(z)} P'_{t'} \tag{4}$$

having dropped $P'_{z'z'}$. The right hand side can be separated into three distinct operations. The first two are merely our diffraction term and the shifting term discussed earlier (Schultz, SEP 5, p. 51-53), but now with $\bar{v} = \bar{v}(z)$. The only additional term which we have inherited from the new transformation is

$$P_{zt} = \frac{\bar{v}_z(z)}{2\bar{v}(z)} P_t \tag{5}$$

where we have dropped the primes. Now this equation can be integrated over t to take the form

$$P_z = a(z)\ P \tag{6}$$

which has solutions

$$P(z + \Delta z) = P(z)\ e^{\int_z^{z+\Delta z} a(z)dz} \tag{7}$$

Now,

$$\int_z^{z+\Delta z} a(z)\ dz = \int_z^{z+\Delta z} \frac{1}{2} \frac{\bar{v}_z(z)}{\bar{v}(z)}\ dz = \int_z^{z+\Delta z} \frac{1}{2} \left[\frac{d}{dz}\ \ell n\ \bar{v}(z) \right] dz$$

$$= \ell n \left[\frac{\bar{v}(z+\Delta z)}{\bar{v}(z)} \right]^{1/2}$$

Therefore, equation (7) reduces to

$$P(z+\Delta z) = P(z) \left[\frac{\bar{v}(z+\Delta z)}{\bar{v}(z)} \right]^{1/2} \tag{8}$$

Equation (8) is obviously a transmission coefficient effect, reducing or increasing the wave amplitude with depth. This effect we will choose to ignore, since it is only an amplitude scaling function which varies with depth, not affecting even U - D coupling because it will scale U and D by the same factor.

So, when we drop this term from equation (4) we find that generalization of $\bar{v}$ to $\bar{v}(z)$ is very convenient indeed. All we need do is replace $\bar{v}$ by $\bar{v}(z)$ in both the diffraction and shifting terms.

Let us now look to the slant frame stratified media transformations.

$$x' = x - \int_0^z \tan\theta(z)\, dz \tag{9a}$$

$$z' = z \tag{9b}$$

$$t' = t - \int_0^z \frac{dz \cos\theta(z)}{\bar{v}(z)} - x \frac{\sin\theta(z)}{\bar{v}(z)} \tag{9c}$$

We now want to express (9) in terms of p and $\bar{v}(z)$ only. We have as identities,

$$\sin\theta(z) = p\,\bar{v}(z) \tag{10a}$$

$$\tan\theta(z) = \frac{p\,\bar{v}(z)}{(1-p^2\bar{v}^2(z))^{1/2}} \tag{10b}$$

$$\cos\theta(z) = (1 - p^2\bar{v}^2(z))^{1/2} \tag{10c}$$

giving,

$$x' = x - \int_0^z \frac{p\,\bar{v}(z)\, dz}{(1-p^2\bar{v}^2(z))^{1/2}} \tag{11a}$$

$$z' = z \tag{11b}$$

$$t' = t - \int_0^z \frac{(1-p^2\bar{v}^2(z))^{1/2}}{\bar{v}(z)}\, dz - x\,p \tag{11c}$$

Then,

$$D_x = D'_{x'} - p\, D'_{t'}$$

$$D_t = D'_{t'}$$

$$D_z = D'_{x'} \left(\frac{-p\bar{v}}{(1-p^2\bar{v}^2)^{1/2}} \right) + D'_{z'} + D'_{t'} \left(-\frac{(1-p^2\bar{v}^2)^{1/2}}{\bar{v}} \right)$$

and

$$D_{xx} = D'_{x'x'} - 2\, p\, D'_{x't'} + p^2\, D'_{t't'}$$

$$D_{tt} = D'_{t't'}$$

$$D_{zz} = \frac{p^2\bar{v}^2}{(1-p^2\bar{v}^2)} D'_{x'x'} + D'_{z'z'} + \frac{(1-p^2\bar{v}^2)}{\bar{v}^2} D'_{t't'}$$

$$- \frac{2\, p\, \bar{v}}{(1-p^2\bar{v}^2)^{1/2}} D'_{x'z'} - 2\, \frac{(1-p^2\bar{v}^2)^{1/2}}{\bar{v}} D'_{z't'} + 2\, p\, D'_{x't'}$$

$$+ \left[\frac{\partial}{\partial z} \frac{-p\bar{v}}{(1-p^2\bar{v}^2)^{1/2}} \right] D'_{x'} + \left[\frac{\partial}{\partial z} \frac{-(1-p^2\bar{v}^2)^{1/2}}{\bar{v}} \right] D'_{t'}$$

Now substitution into the wave equation gives

$$D'_{x'x'} - 2\, p\, D'_{x't'} + p^2\, D'_{t't'} + \frac{p^2\bar{v}^2}{(1-p^2\bar{v}^2)^{1/2}} D'_{x'x'}$$

$$+ D'_{z'z'} + \frac{(1-p^2\bar{v}^2)}{\bar{v}^2} D'_{t't'} + \frac{2\, p\, \bar{v}}{(1-p^2\bar{v}^2)^{1/2}} D'_{x'z'}$$

$$- \frac{2(1-p^2\bar{v}^2)^{1/2}}{\bar{v}} D'_{z't'} + 2\, p\, D'_{x't'} + \left[\frac{\partial}{\partial z} \frac{-p\bar{v}}{(1-p^2\bar{v}^2)^{1/2}} \right] D'_{x'}$$

$$+ \left[\frac{\partial}{\partial z} \frac{-(1-p^2\bar{v}^2)^{1/2}}{\bar{v}} \right] D'_{t'} = \frac{1}{\bar{v}^2} D'_{t't'} \qquad (12)$$

We now drop the $D'_{z'z'}$ and notice that the $D'_{x't'}$ terms add to zero. In addition, we drop the $D'_{x'z'}$ term because it has been shown to be small in previous work (see for example, Doherty SEP 4, p. 45). Collecting all remaining terms gives

$$
\begin{aligned}
D'_{z't'} &= \frac{\bar{v}(z)}{2} \left(1 - p^2\bar{v}^2(x) \right)^{-3/2} D'_{x'x'} \\
&\quad + \frac{\bar{v}(z)}{2(1-p^2\bar{v}^2(z))^{1/2}} \left(\frac{1}{\bar{v}^2(z)} - \frac{1}{\tilde{v}^2(x,z)} \right) D'_{t't'} \\
&\quad + \frac{p\bar{v}(z)\,\bar{v}_z(z)}{2(1-p^2\bar{v}^2(z))^2} D'_{x'} + \left[\frac{p^2\bar{v}(z)\,\bar{v}_z(z)}{2(1-p^2\bar{v}^2(z))} + \frac{\bar{v}_z(z)}{2\bar{v}(z)} \right] D'_{t'}
\end{aligned}
\tag{13}
$$

The first two terms give new expressions for the diffraction and shifting. Although the coefficients are z-dependent, the form is the same as before with $\bar{v}$ constant. Let us now look at the last two terms.

$$
D'_{z't'} = \frac{p\,\bar{v}(z)\bar{v}_z(z)}{2(1-p^2\bar{v}^2(z))^2} D'_{x'} + \left[\frac{p^2\bar{v}(z)\,\bar{v}_z(z)}{2(1-p^2\bar{v}^2(z))} + \frac{\bar{v}_z(z)}{2\bar{v}(z)} \right] D'_{t'}
$$

We separate the operations

$$
D'_{z't'} = \frac{p\,\bar{v}(z)\,\bar{v}_z(z)}{2(1-p^2\bar{v}^2(z))^2} D'_{x'} \tag{14a}
$$

$$
D'_{z't'} = \left[\frac{p^2\bar{v}(z)\,\bar{v}_z(z)}{2(1-p^2\bar{v}^2(z))} + \frac{\bar{v}_z(z)}{2\bar{v}(z)} \right] D'_{t'} \tag{14b}
$$

To equation (14a) let us transform out the t and x dependence, and to equation (14b) let us integrate with respect to t. Then combining the equations we have

$$D'_{z'} = \{ \frac{p\,\bar{v}_z(z)}{2(1-p^2\bar{v}^2)^2} \left(\frac{\bar{v}(z)\,k_{x'}}{\omega}\right) + \frac{p^2\bar{v}(z)\bar{v}_z(z)}{2(1-p^2\bar{v}^2(z))} + \frac{\bar{v}_z(z)}{2\bar{v}(z)} \} D' \qquad (15)$$

We notice the form of (15) is similar to the form of (5); indeed for $p=0$ it reduces exactly to equation (5). We are led then to interpret (15) as a transmission term.

Without permuting the terms, we can rewrite (15) in the form,

$$D'_{z'} = [\, a_3(z,\, p,\, \frac{\bar{v}\,k_{x'}}{\omega}) + a_2(z,\, p) + a_1(z) \,]\, D' \qquad (16)$$

The factor a_1 is a function only of z and will be non-zero for even the simplest case, being vertical incidence ($p = \theta = 0$) of a plane wavefront onto an interface. This is shown in Figure 1.

The factor a_2 is a function of both z and p , and will be non-zero only for $p \neq 0$. Figure 2 shows the simplest case of a non-zero a_2 : a plane wavefront incident on an interface at an angle given by the slant frame transformation (i.e., θ) .

The factor a_3 is a function of z, p, and $k_{x'}$. In addition to a sensitivity to z and p , as with a_2 , it is also sensitive to a non-zero $k_{x'}$ (i.e., a non-zero $\partial/\partial_{x'}$). The incident wavefront of Figure 2 follows a curve $x' = \text{constant}$, so that $\partial/\partial_{x'} = k_{x'} = 0$ for this case. The factor a_3 is therefore sensitive to deviations from the ideal incident angle, θ . Figure 3 shows a simple case of a distorted (i.e., non-planar) wavefront impinging on an interface for which case a_3 is non-zero.

At this point a disclaimer should be made in regard to the interpretation of ($a_1 + a_2 + a_3$) in equation (16) as a transmission coefficient. It will be true only to the extent that the gradients of density are everywhere zero; any non-zero density gradient will change the form of equation (15). However, we will not reformulate the problem to include the density because we do not foresee any advantages to including the transmission terms (equation (15)) in our difference formulation of the transformed wave equation (13). These terms may be considered amplitude correction terms, and although they may be interesting in some modeling cases (for example, modeling the dim regions below bright spots), they will be ignored for our applications.

Dropping these transmission terms leaves us an equation of the form

$$D'_{z't'} = \frac{\bar{v}(z)}{2} \left(1 - p^2\bar{v}^2(z)\right)^{-3/2} D'_{x'x'} + \frac{\bar{v}(z)}{2(1-p^2\bar{v}^2(z))^{1/2}} \left[\frac{1}{\bar{v}^2(z)} - \frac{1}{\tilde{v}^2(x,z)} \right] D'_{t't'} \qquad (17)$$

which includes only the two terms with which we are most familiar. The only correction that has been made to the constant velocity slant frame transformation equations is the substitution of $\bar{v}(z)$ for $\bar{v}$, making the generalization to a vertically stratified migration velocity extremely convenient.