Slant Midpoint Coordinates

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Suppose shots were fired in rapid succession so as to provide a slanted downgoing plane wave for earth illumination. Such a situation may be synthesized from conventional data by means of the common geophone slant stack. This stack differs from the industry common reflection point stack in two important respects.

(1) First, in the slant stack the moveout function is linear with offset and independent of time. Thus, the usual industry hyperboloidal moveout before stack has a "focusing" advantage.

(2) Second, the industry stack is done at a common midpoint, not at a common geophone. Thus, the industry stack has the advantage that for layered media it does not "smear" subsurface information.

The above advantages of the industry stack are degraded where the earth is not horizontally stratified. This degradation appears as processing artifacts which are difficult to accomodate by subsequent processing. For common geophone slant stack, however, the seismic section is also a seismic wave from a real experiment. Any "artifact" in this stack must be manageable with the wave equation.

Our most immediate goals are to show that we can predict diffracted multiples and estimate rapid but smooth horizontal velocity variations on slant stacked data. If we achieve these goals we can then turn our attention to trying to recover some of the advantages of the common midpoint NMO stack. My plan in this section is to begin with a precise definition of up- and downgoing slanted wave common geophone stacks. Then I will show how I was forced to make a new and unfamiliar approximation when I tried to extend this work to common midpoint stacks.

Take velocity $v=1$. The transformation pair of interest for downgoing waves is

$$\begin{bmatrix} t \\ s \\ g \\ z \end{bmatrix} = \begin{bmatrix} 1 & \sin & 0 & \cos \\ 0 & -1 & +1 & +\tan \\ 0 & 0 & 1 & +\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ f' \\ g' \\ z' \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} t' \\ f' \\ g' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & \sin & -\sin & -\cos \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 1 & -\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ s \\ g \\ z \end{bmatrix} \tag{2}$$

For waves going up we change the sign of z and z' getting

$$\begin{bmatrix} t \\ s \\ g \\ z \end{bmatrix} = \begin{bmatrix} 1 & \sin & 0 & -\cos \\ 0 & -1 & +1 & -\tan \\ 0 & 0 & 1 & -\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'' \\ f'' \\ g'' \\ z'' \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} t'' \\ f'' \\ g'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & \sin & -\sin & \cos \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ s \\ g \\ z \end{bmatrix} \tag{4}$$

I cannot assert that this is the only possible transformation which will meet our needs. Indeed from SEP-5, p. 20, it seems that in homogeneous media we may have a wider choice. However, once the zeros are placed below the main diagonal in (2) we can see how all the other elements have been forced to take their stated values.

First of all, the zero in the (2,4) position in (2) and (4) enables f' to equal f''. For offset f to have the same definition in either frame will later be seen to be necessary when we integrate the wave equation over offset. With these zeros we use the chain rule of partial differentiation which gives

$$P(t, s, g, z) = P'(t', f, g', z') \tag{6a}$$

$$P_g = P'_{t'} t'_g + P'_f f_g + P'_{g'} g'_g + P'_{z'} z'_g \tag{6b}$$

$$P_{gg} = (t'_g \partial_{t'} + f_g \partial_f + \partial_{g'})^2 P' \tag{6c}$$

$$P_z = P'_{t'} t'_z + P'_f f_z + P'_{g'} g'_z + P'_{z'} z'_z \tag{6d}$$

$$P_{zz} = (t'_z \partial_{t'} + g'_z \partial_{g'} + \partial_z)^2 P' \tag{6e}$$

$$P_t = P'_{t'} t'_t + P'_f f_t + P'_{g'} g'_t + P'_{z'} z'_t \tag{6f}$$

$$P_{tt} = \partial_{t't'} P' \tag{6g}$$

The usual magnitude condition

$$\partial_{t'} >> \partial_{g'} >> \partial_{z'} \tag{7}$$

means that we must insure the vanishing of the coefficients of $\partial_{t't'}$ and $\partial_{t'g'}$ so that we will finish with the $\partial_{z't'}$ and $\partial_{g'g'}$ terms dominating the final equation.

For the coefficient of $\partial_{t't'}$ from (6c, e, and g) and the wave equation $P_{gg} + P_{zz} - P_{tt} = 0$ we get

$$(t'_g)^2 + (t'_z)^2 - 1 = 0 \qquad (8)$$

This condition is satisfied by choice of an angle parameter θ and choosing t'_g as $-\sin\theta$ and t'_z as $-\cos\theta$ in (2) . For the coefficient of $\partial_{t'g'}$ from (6c,e) we get

$$t'_g + t'_z \, g'_z = 0 \qquad (9)$$

from which we conclude that $g'_z = -\tan\theta$. That accounts for all the elements in (2). The transformation (1) was found by matrix inversion of (2) and the transformations (3) and (4) were found by sign change of the z axis.

From (6) we obtain the upcoming wave operator and from "Coupled Slanted Waves, Monochromatic Derivation" on page 31 we obtain the result

$$[(\partial_f + \partial_{g''})^2 - \cos \partial_{t''z''} + \tan^2 \partial_{g''g''}] \, U'' = c(g,z) \, D_t \qquad (10)$$

Following "Coupled Slanted Beams, Equations for Multiples Program", page 33 we plan to express all variables in the upcoming frame. We need an equation for downgoing variables in terms of upgoing variables which we obtain by substituting (3) into (2).

$$\begin{bmatrix} t' \\ f' \\ g' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & \sin & -\sin & -\cos \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 1 & -\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sin & 0 & -\cos \\ 0 & -1 & 1 & -\tan \\ 0 & 0 & 1 & -\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'' \\ f'' \\ g'' \\ z'' \end{bmatrix}$$

$$\begin{bmatrix} t' \\ f' \\ g' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2\cos \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'' \\ f'' \\ g'' \\ z'' \end{bmatrix} \qquad (11)$$

Inserting into (10) and expressing D' as a function of upcoming variables as earlier, we get

$$\partial^{t''} [(\partial_f + \partial_{g''})^2 - \cos \partial_{t''z''} + \tan^2 \partial_{g''g''}] U'' (t'', f'', g'', z'') =$$
$$= c (g'' - z'' \tan, z) \quad D' (t'' - 2z'' \cos, f'', g'' - 2z'' \tan, z'') \qquad (12)$$

Because we have carefully insured that $f'' = f' = f$ we can easily integrate this equation over f. The integral of U_f and U_{ff} will vanish by end conditions. Because the reflection coefficient c is independent of offset it comes out of the integral. (Here is the hangup when we attempt to do the same thing with common midpoint sections.) Using the overbar to denote integral over f and dropping double primes we are left with

$$(- \cos \partial_z + \cos^{-2} \partial^t_{gg}) \bar{U} = c(g-z\tan,z)\bar{D} (t - 2z\cos, g - 2z\tan, z) \qquad (13)$$

By way of further clarifying the nature of the difficulty with common midpoint coordinates the following common midpoint transformations were devised:

For upcoming waves:

$$\begin{bmatrix} t \\ g \\ s \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2\sin & -\cos \\ 0 & 1 & 1 & \frac{1}{2}\tan \\ 0 & 1 & -1 & \frac{1}{2}\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}''$$

$$\begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}'' \quad \begin{bmatrix} 1 & \sin & -\sin & \cos \\ 0 & 1/2 & 1/2 & -\frac{1}{2}\tan \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ g \\ s \\ z \end{bmatrix}$$

For downgoing waves:

$$\begin{bmatrix} t \\ g \\ s \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2\sin & \cos \\ 0 & 1 & 1 & -\frac{1}{2}\tan \\ 0 & 1 & -1 & -\frac{1}{2}\tan \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}'$$

$$\begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}' = \begin{bmatrix} 1 & \sin & -\sin & -\cos \\ 0 & 1/2 & 1/2 & \frac{1}{2}\tan \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ g \\ s \\ z \end{bmatrix}$$

For conversion of up to down:

$$\begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}' = \begin{bmatrix} 1 & \sin & -\sin & -\cos \\ 0 & 1/2 & 1/2 & \frac{1}{2}\tan \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2\sin & -\cos \\ 0 & 1 & 1 & \frac{1}{2}\tan \\ 0 & 1 & -1 & \frac{1}{2}\tan \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}''$$

$$\begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 & -2\cos \\ 0 & 1 & 0 & \tan \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ y \\ h \\ z \end{bmatrix}''$$

$$\partial_z U'' = \partial^{d''}_{y''y''} U'' - c(g,z)\, D'(d', y', h', z)$$

$$- c\left(y'' + h'' + \frac{z}{2}\tan\theta, z''\right) D'(d''-2z\cos, y+z\tan, h'', z'')$$

The problem with this is that the upcoming wave equation looks like

$$\left(\partial_z + \partial^{d''}_{y''y''}\right) U'' = c(g,z)\, D'(d', y', h', z')$$

$$= c\left(y'' + h'' + \frac{z''}{2}\tan, z''\right) D'(d'' - 2z''\cos, y'' + z''\tan, h'', z'')$$

When we integrate this equation over the half offset h'', we want to bring the integral through the reflection coefficient. It is obviously valid in a stratified earth. In a non-stratified earth we must argue that c is changing slowly compared to the size of the Fresnel zone of D'. We are at a fork in the road where we could choose to determine the range of validity of this approximation, but instead what we do is to set aside our efforts to work with common midpoint stacks.