Coupled Slanted Beams, Equations for Multiples Program

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For downgoing waves the slant transformation and its inverse are

$$\begin{bmatrix} t' \\ x' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -\sin & -\cos \\ 0 & 1 & -\tan \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ z \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} t \\ x \\ z \end{bmatrix} = \begin{bmatrix} 1 & \sin & 1/\cos \\ 0 & 1 & \tan \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t' \\ x' \\ z' \end{bmatrix} \tag{2}$$

The way I remember how to get the upcoming wave transforms is to change the sign of the z-axis in equations (1) and (2). This is done by changing the sign of each element in the third column, then changing the sign of each element in the third row. The diagonal element thus changes twice. This gives

$$\begin{bmatrix} t'' \\ x'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & -\sin & \cos \\ 0 & 1 & \tan \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ z \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} t \\ x \\ z \end{bmatrix} = \begin{bmatrix} 1 & \sin & -1/\cos \\ 0 & 1 & -\tan \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'' \\ x'' \\ z'' \end{bmatrix} \tag{4}$$

The transformation from upcoming wave coordinates to downgoing wave coordinates is found by inserting (4) into (1), namely

$$\begin{bmatrix} t' \\ x' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -\sin & -\cos \\ & 1 & -\tan \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & \sin & -1/\cos \\ & 1 & -\tan \\ & & 1 \end{bmatrix} \begin{bmatrix} t'' \\ x'' \\ z'' \end{bmatrix}$$

$$\begin{bmatrix} t' \\ x' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2\cos \\ 0 & 1 & -2\tan \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t'' \\ x'' \\ z'' \end{bmatrix} \qquad (5)$$

Define reflection coefficient density as

$$c(x,z) = \frac{1}{2} \frac{(\rho v)_z}{\rho v} \qquad (6)$$

Although the prime system is natural for the downgoing wave D, the double prime system is natural for the upcoming wave U, and the reflection coefficients are naturally expressed in unprimed coordinates, we may express each in any coordinate system we wish.

$$D(x,z,t) = D'(x', z', t') = D''(x'', z'', t'') \qquad (7a)$$

$$U(x, z, t) = U'(x', z', t') = U''(x'', z'', t'') \qquad (7b)$$

$$c(x, z) = c'(x', z', t') = c''(x'', z'', t'') \qquad (7c)$$

By inserting (2) or (4) into the left side of (7c) we discover that the time invariance of the earth in unprimed coordinates luckily has a simple form in primed coordinates, namely, we can replace (7c) by

$$c(x, z) = c'(x', z') = c''(x'', z'') \qquad (8)$$

Now we return to "Coupled Slanted Waves, Monochromatic Derivation" and extract the final result, equation (13). Neglecting the angular dependence of reflection coefficients term and using present notation we have

$$\partial_{z''} U''(t'',x'',z'') = -\frac{v}{2\cos^3\theta}\,\partial^{t''}_{x''x''} U''(t'',x'',z'') - c(x,z)\,D(t,x,z) \tag{9}$$

For programming purposes we want all the independent variables in (9) to be double prime variables, but we want to express the downgoing wave in its natural computational form, namely D' . Using (7) and (5) we are able to get the final equations

$$\partial_{z''}U''(t'',x'',z'') = \frac{-v}{2\cos^3\theta}\,\partial^{t''}_{x''x''}U''(t'',x'',z'') - c''(x'',z'')\,D'\!\left(t''-\frac{2z''}{v}\cos\theta,\, x''-2z''\tan\theta,\, z''\right) \tag{10a}$$

$$\partial_{z'} D'(t',x',z') = \frac{v}{2\cos^3\theta}\,\partial^{t'}_{x'x'} D'(t',x',z') \tag{10b}$$