III-C. EXTENDING THE MULTICHANNEL AUTOCORRELATION FUNCTION

In studying the general extension of the single channel autocorrelation function, we saw that if the autocorrelation matrix became singular, i.e., if the mean square error went to zero, then the rest of the autocorrelation function was resolved. In the multichannel case, the situation is more complex when the determinant of the mean square error matrix vanishes. When this happens, at least one of the time series can be generated by a multichannel recursive filter operating on the past of the time series in question and the present and past of the other time series. Such channels can be temporarily eliminated from consideration and the extension of the autocorrelation function of the remaining channels can be continued separately. The extension of the complete autocorrelation function can be recreated by using the deterministic relationship that the eliminated channels have with the other channels.

All of the various theorems discussed in the single channel case have their counterparts in the multichannel situation. Since almost all of these theorems are based on the Fundamental Autocorrelation Matrix Theorem, we shall restrict our attention to its proof. If the need arises, the reader can develop the various multichannel theorems, together with the complexities caused by the vanishing of the determinant of the mean square error matrix.

1. The Fundamental Multichannel Autocorrelation Matrix Theorem

Given a set of M by M complex matrices, $R(n)$, $|n| \leq N$, where $R(n) = R^{\dagger}(-n)$, then these matrices are the beginning of an M channel multichannel autocorrelation function if and only if the block Toeplitz matrix

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix}$$

is non-negative definite.

By looking at the single channel case, it is clear that such a matrix must be non-negative definite. Thus, we shall be concerned with proving the sufficiency part of the theorem. Because the strictly non-negative definite situation is a limiting case of the positive definite case, it is only necessary to prove the theorem when the block Toeplitz matrix is positive definite. We shall do this by constructing a multichannel spectrum whose autocorrelation values agree with the given $R(n)$, $|n| \leq N$. In particular, the constructed spectrum will be the maximum entropy spectrum. The proof involves first showing that the multichannel prediction error filter is minimum phase and then algebraically demonstrating that the corresponding maximum entropy spectrum agrees with the given autocorrelation values.

2. The Multichannel Minimum Phase Theorem

One definition of a minimum phase filter is that it is a physically realizable filter whose inverse is also physically realizable (and stable). In the single channel case, for an Nth order, physically realizable filter, $1 + a_1 z + \ldots + a_N z^N$, to be minimum phase, it is necessary that $[1 + a_1 z + \ldots + a_N z^N]^{-1}$ have a Taylor series expansion which converges on the unit circle. This will be true if and only if $[1 + a_1 z + \ldots a_N z^N]^{-1}$ has no poles on or inside the unit circle, which means that all N roots of $1 + a_1 z + \ldots + a_N z^N$ must lie outside the unit circle.

A minimum phase multichannel filter is similarly defined. That is, it is a physically realizable filter whose inverse is physically realizable. This means that for an Nth order, physically realizable filter of the form, $F(z) = I + F_1 z + \ldots + F_N z^N$, to be minimum phase, it is necessary that $F^{-1}(z)$ have a convergent matrix Taylor series expansion on the unit circle. As an example of what $F^{-1}(z)$ looks like, let us consider a two channel case and write

$$F(z) = \begin{bmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{bmatrix} .$$

Then

$$F^{-1}(z) = \frac{1}{\det[F(z)]} \begin{bmatrix} F_{22}(z) & -F_{12}(z) \\ -F_{21}(z) & F_{11}(z) \end{bmatrix}$$

From this, it is clear in general that for $F^{-1}(z)$ to be physically realizable, it is necessary and sufficient for the determinant of

$F(z)$ to have all of its zeros outside the unit circle.

We shall now prove that if the Nth order multichannel Toeplitz matrix is positive definite, then the corresponding Nth order prediction error filter is minimum phase. That is, if

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_N \end{bmatrix} = \begin{bmatrix} P_N \\ 0 \\ \\ 0 \end{bmatrix} , \qquad \text{(III-25)}$$

where the Toeplitz matrix is positive definite, then the determinant of $I + F_1 z + \ldots + F_N z^N$ has all of its roots outside the unit circle.

The proof begins by augmenting (III-25) to get

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) & R(-N-1) \\ R(1) & R(0) & & R(1-N) & R(-N) \\ & & & & \\ R(N) & R(N-1) & & R(0) & R(-1) \\ R(N+1) & R(N) & & R(1) & R(0) \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_N \\ 0 \end{bmatrix} = \begin{bmatrix} P_N \\ 0 \\ \\ 0 \\ 0 \end{bmatrix} . \qquad \text{(III-26)}$$

For this equation to be valid, it is only necessary that

$$R(N+1) = - \sum_{n=1}^{N} R(N+1-n) F_n .$$

In addition, this N+1 th order Toeplitz matrix, R_{N+1} , will also be positive definite. This can be verified by letting

$$Q = \begin{bmatrix} I & 0 & 0 & 0 \\ F_1 & I & 0 & 0 \\ & & & \\ F_N & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} ,$$

and noting that

$$Q^{\dagger} R_{N+1} Q = \begin{bmatrix} P_N & 0 & 0 & 0 \\ 0 & R(0) & R(1-N) & R(-N) \\ & & & \\ 0 & R(N-1) & R(0) & R(-1) \\ 0 & R(N) & R(1) & R(0) \end{bmatrix} .$$

Since P_N is positive definite, so will be R_{N+1} . It is also clear from these equations that

$$\det [R_{N+1}] = \prod_{n=0}^{N+1} |P_n| ,$$

a fact which we shall use a little later on.

Using the above procedure, we can continue to augment the Toeplitz matrix indefinitely and the enlarged matrix will always be positive definite. Our extension of the R matrices follows the feedback equation

$$\sum_{n=0}^{N} R(s-n) F_n = 0 , \quad (s > 0) , \qquad \text{(III-27)}$$

where $F_0 \equiv I$.

This equation can be extended to all values of s . In particular, for $s=0$, we have

$$\sum_{n=0}^{N} R(-n)\, F_n = P_N \ , \qquad \text{(III-28)}$$

and for $s < 0$, one can calculate the M by M matrices, $A(s)$, by

$$\sum_{n=0}^{N} R(s-n)\, F_n = A(s) \ , \quad (s < 0) \ . \qquad \text{(III-29)}$$

These equations can be written in z transform form as

$$\Big[\sum_{n=-\infty}^{+\infty} R(n)\, z^n \Big]\, F(z) = P_N + \sum_{n=-1}^{-\infty} A(n)\, z^n$$

or

$$\sum_{n=-\infty}^{+\infty} R(n)\, z^n = \Big[P_N + \sum_{n=-1}^{-\infty} A(n)\, z^n \Big]\, F^{-1}(z) \ . \qquad \text{(III-30)}$$

This equation can be interpreted as a multichannel filtering operation where the multichannel input $P_N + \Sigma A(n) z^n$ is filtered by the inverse of the multichannel prediction error filter to get the auto-correlation function $\Sigma R(n) z^n$. The input becomes zero after $t=0$, but the output continues on indefinitely. To understand this, we see that a typical z-transform term of $F^{-1}(z)$ is of the form

$$\frac{\text{polynomial in } z}{\det F(z)} \ ,$$

and that the extension of R will consist of linear combinations of such terms. If the $\det F(z)$ has a zero inside the unit circle, then that zero will create an exponentially growing term in the R extension unless the coefficient weighting that term is zero. To show that the coefficient could not be zero, let us assume that w is a root of $\det F(z)$ and that $e = \{e_1\ e_2\ \dots\ e_M\}$ is an eigenvector of $F(z)$, i.e.,

$$\{ e_1 \; e_2 \; \dots \; e_M \} F(w) = e\,F(w) = \{ 0 \; 0 \; \dots \; 0 \} .$$

Then

$$[\, e \quad ew \quad ew^2 \quad ew^N \,] \begin{bmatrix} I \\ F_1 \\ \\ F_N \end{bmatrix} = \{ 0 \; 0 \; \dots \; 0 \} . \qquad \text{(III-31)}$$

Thus $[\, e \;\; ew \;\; ew^2 \dots ew^N \,]$ is one of the vectors that is perpendicular to the M column vectors of the F matrix. Since these column vectors are independent (the identity matrix assures this) and $MN + M$ tall, there will be exactly MN linearly independent vectors perpendicular to the M columns. Looking at (III-25), we see that the bottom MN rows are all perpendicular to the F matrix. Furthermore, because of the positive definite assumption, these rows are linearly independent, and thus cannot all be perpendicular to $[\, e \;\; ew \;\; ew^2 \dots ew^N \,]$. Thus if the $\det F(z)$ has a zero inside the unit circle, the extension of the R matrices will contain an exponentially growing term. However, because all enlarged R matrices are positive definite, we cannot have arbitrarily large off diagonal terms. Thus we conclude that all zeros of $\det F(z)$ must lie on or outside the unit circle. We shall now show that $\det F(z)$ cannot have a zero on the unit circle.

Suppose w is a zero of unit magnitude. Then, using (III-31), we see that

$$\left[\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} - \alpha \begin{bmatrix} e^{\dagger} \\ e^{\dagger}\, w^{-1} \\ \\ e^{\dagger}\, w^{-1} \end{bmatrix} [e \quad ew \quad\quad ew^{N}]\right] \begin{bmatrix} I \\ F_1 \\ \\ F_N \end{bmatrix} = \begin{bmatrix} P_N \\ 0 \\ \\ 0 \end{bmatrix}$$

for any value of the real constant, α . We note that the dyadic matrix is a block Toeplitz matrix, and thus combining it with the R matrix still gives us a block Toeplitz matrix. If α is allowed to increase from zero, the determinant of the Toeplitz matrix can be made as small as one wishes. However, we have already shown that if the F_n and P_N matrices are the solution for a positive definite block Toeplitz matrix, then the determinant of that matrix is given by $\prod_{n=0}^{N} |P_n|$. But since $|P_{n+1}| \leq |P_n|$, the determinant must be greater than or equal to $|P_N|^{N+1}$. Since $|P_N| > 0$, we see that det F(z) cannot have a zero on the unit circle.

3. The Multichannel Maximum Entropy Extension

Suppose we have $R(n)$, $|n| \lesssim N$, of a positive definite multichannel autocorrelation function and we solve (III-25) for the Nth order forward prediction error filter. If we now choose the zero extension of the reflection coefficient sequence, starting with C_{N+1} , then the corresponding extension of the autocorrelation function will be given by the convolutional matrix feedback operation of

$$R(s) = -\sum_{n=1}^{N} R(s-n)\, F_n \ , \quad s > N \ , \text{ or}$$

$$\sum_{n=0}^{N} R(s-n)\, F_n = 0 \ , \quad s > N \ , \quad F_0 \equiv I \ . \qquad \text{(III-32)}$$

Let us define the function $H(z)$ as

$$H(z) = R(0)/2 + R(1)z + R(2)z^2 + \dots \ . \qquad \text{(III-33)}$$

It is clear that $H(z) + H^{\dagger}(z^{-1})$ is the z transform of the multichannel autocorrelation function. If we convolve $H(z)$ with the Nth order prediction error filter, $F(z)$, the result must be of the form,

$$H(z)\,F(z) = \frac{R(0)}{2}\,[\,I + \bar{F}_1\,z + \dots \bar{F}_N\,z^N\,] = \frac{R(0)}{2}\,\bar{F}(z) \ , \qquad \text{(III-34)}$$

since (III-32) tells us that $\bar{F}_s = 0$ for $s > N$. We see from (III-34) that

$$H(z) = \frac{R(0)}{2}\,\bar{F}(z)\,F^{-1}(z) \qquad \text{(III-35)}$$

and that $H(z)$ is a legitimate matrix Taylor series since $F(z)$ is minimum phase.

We now derive the relationship between $F(z)$ and $\bar{F}(z)$ by induction. At the same time, we shall develop the relationship between the backward prediction error quantities $B(z)$ and $\bar{B}(z)$.

Suppose that

$$\begin{bmatrix} R(0)/2 & 0 & & 0 \\ R(1) & R(0)/2 & & 0 \\ & & & \\ R(N) & R(N-1) & & R(0)/2 \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_N \end{bmatrix} = \begin{bmatrix} R(0)/2 & 0 & & 0 \\ 0 & R(0)/2 & & 0 \\ & & & \\ 0 & 0 & & R(0)/2 \end{bmatrix} \begin{bmatrix} I \\ \bar{F}_1 \\ \\ \bar{F}_N \end{bmatrix} \tag{III-36}$$

and

$$\begin{bmatrix} R(0)/2 & R(-1) & & R(-N) \\ 0 & R(0)/2 & & R(1-N) \\ & & & \\ 0 & 0 & & R(0)/2 \end{bmatrix} \begin{bmatrix} B_N \\ B_{N-1} \\ \\ I \end{bmatrix} = \begin{bmatrix} R(0)/2 & 0 & & 0 \\ 0 & R(0)/2 & & 0 \\ & & & \\ 0 & 0 & & R(0)/2 \end{bmatrix} \begin{bmatrix} \bar{B}_N \\ \bar{B}_{N-1} \\ \\ I \end{bmatrix} \tag{III-37}$$

are true. Then since

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_N \end{bmatrix} = \begin{bmatrix} P_N \\ 0 \\ \\ 0 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} \begin{bmatrix} B_N \\ B_{N-1} \\ \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ P'_N \end{bmatrix} , \text{ we have}$$

$$\begin{bmatrix} R(0)/2 & R(-1) & & R(-N) \\ 0 & R(0)/2 & & R(1-N) \\ & & & \\ 0 & 0 & & R(0)/2 \end{bmatrix} \begin{bmatrix} I \\ F_1 \\ \\ F_N \end{bmatrix} = \begin{bmatrix} P_N \\ 0 \\ \\ 0 \end{bmatrix} - \begin{bmatrix} R(0)/2 & 0 & 0 \\ 0 & R(0)/2 & 0 \\ & & \\ 0 & 0 & R(0)/2 \end{bmatrix} \begin{bmatrix} I \\ \bar{F}_1 \\ \\ \bar{F}_N \end{bmatrix} \tag{III-38}$$

and

$$\begin{bmatrix} R(0)/2 & 0 & & 0 \\ R(1) & R(0)/2 & & 0 \\ & & & \\ R(N) & R(N-1) & & R(0)/2 \end{bmatrix} \begin{bmatrix} B_N \\ B_{N-1} \\ \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ P_N' \end{bmatrix} - \begin{bmatrix} R(0)/2 & 0 & 0 \\ 0 & R(0)/2 & 0 \\ & & \\ 0 & 0 & R(0)/2 \end{bmatrix} \begin{bmatrix} \bar{B}_N \\ \bar{B}_{N-1} \\ \\ I \end{bmatrix} . \tag{III-39}$$

Going to the $N+1$ case and using (III-36) and (III-39), we can write for the forward prediction error filter that

$$\begin{bmatrix} R(0)/2 & 0 & & 0 & 0 \\ R(1) & R(0)/2 & & 0 & 0 \\ & & & & \\ R(N) & R(N-1) & & R(0)/2 & 0 \\ R(N+1) & R(N) & & R(1) & R(0)/2 \end{bmatrix} \left[\begin{bmatrix} I \\ F_1 \\ \\ F_N \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ B_N \\ \\ B_1 \\ I \end{bmatrix} C_{N+1} \right] =$$

$$\begin{bmatrix} R(0)/2 & 0 & 0 & 0 \\ 0 & R(0)/2 & 0 & 0 \\ & & & \\ 0 & 0 & R(0)/2 & 0 \\ 0 & 0 & 0 & R(0)/2 \end{bmatrix} \left[\begin{bmatrix} I \\ \bar{F}_1 \\ \\ \bar{F}_N \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{B}_N \\ \\ \bar{B}_1 \\ I \end{bmatrix} C_{N+1} \right] + \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ \Delta_{N+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ P_N' \end{bmatrix} C_{N+1} =$$

$$\begin{bmatrix} R(0)/2 & 0 & & 0 & 0 \\ 0 & R(0)/2 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & R(0)/2 & 0 \\ 0 & 0 & & 0 & R(0)/2 \end{bmatrix} \left[\begin{bmatrix} I \\ \bar{F}_1 \\ \vdots \\ \bar{F}_N \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \bar{B}_N \\ \vdots \\ \bar{B}_1 \\ I \end{bmatrix} C_{N+1} \right] , \qquad \text{(III-40)}$$

since $\Delta_{N+1} + P_N' C_{N+1} = 0$.

Likewise, from (III-37) and (III-38) , we get

$$\begin{bmatrix} R(0)/2 & R(-1) & & R(-N) & R(-1-N) \\ 0 & R(0)/2 & & R(1-N) & R(-N) \\ & & \ddots & & \\ 0 & 0 & & R(0)/2 & R(-1) \\ 0 & 0 & & 0 & R(0)/2 \end{bmatrix} \left[\begin{bmatrix} 0 \\ B_N \\ \vdots \\ B_1 \\ I \end{bmatrix} + \begin{bmatrix} I \\ F_1 \\ \vdots \\ F_N \\ 0 \end{bmatrix} C_N' \right] =$$

$$\begin{bmatrix} R(0)/2 & 0 & & 0 & 0 \\ 0 & R(0)/2 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & R(0)/2 & 0 \\ 0 & 0 & & 0 & R(0)/2 \end{bmatrix} \left[\begin{bmatrix} 0 \\ \bar{B}_N \\ \vdots \\ \bar{B}_1 \\ I \end{bmatrix} - \begin{bmatrix} I \\ \bar{F}_1 \\ \vdots \\ \bar{F}_N \\ 0 \end{bmatrix} C_N' \right] + \begin{bmatrix} \Delta_{N+1}^{\dagger} \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} P_N \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} C_N'$$

$$= \begin{bmatrix} R(0)/2 & 0 & & 0 & 0 \\ 0 & R(0)/2 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & R(0)/2 & 0 \\ 0 & 0 & & 0 & R(0)/2 \end{bmatrix} \left[\begin{bmatrix} 0 \\ \bar{B}_N \\ \vdots \\ \bar{B}_1 \\ I \end{bmatrix} - \begin{bmatrix} I \\ \bar{F}_1 \\ \vdots \\ \bar{F}_N \\ 0 \end{bmatrix} C_N' \right] , \qquad \text{(III-41)}$$

since $\Delta_{N+1}^{\dagger} + P_N C_N' = 0$.

Thus our assumptions for the Nth case of (III-36) and (III-37) lead to the corresponding equations (III-40) and (III-41) for N+1th case. This means that $\bar{F}(z)$ and $\bar{B}(z)$ are built up from the reflection coefficients in exactly the same manner as are F(z) and B(z) except that the negatives of the reflection coefficients are used.

We will now prove that the extension of the autocorrelation function given by (III-32) does indeed correspond to the maximum entropy spectrum by showing that

$$H(z) + H^{\dagger}(z^{-1}) = F^{-1\dagger}(z^{-1})\, P_N\, F^{-1}(z) \quad . \qquad \text{(III-42)}$$

Actually, by using (III-35), we see that

$$H(z) + H^{\dagger}(z^{-1}) = \frac{R(0)}{2}\bar{F}(z)\, F^{-1}(z) + F^{-1\dagger}(z^{-1})\, \bar{F}^{\dagger}(z^{-1})\, \frac{R(0)}{2}$$

$$= \frac{1}{2} F^{-1\dagger}(z^{-1})[\, F^{\dagger}(z^{-1})\, R(0)\, \bar{F}(z) + \bar{F}^{\dagger}(z^{-1})\, R(0)\, F(z)\,]\, F^{-1}(z) \; . \qquad \text{(III-43)}$$

Thus, (III-42) will be true if

$$2\, P_N = [\, F^{\dagger}(z^{-1})\, R(0)\, \bar{F}(z) + \bar{F}^{\dagger}(z^{-1})\, R(0)\, F(z)\,] \qquad \text{(III-44)}$$

is true. Our proof will be by induction and will be illustrated by going from N-1 to N .

From (III-16), we see that if F(z), $\bar{F}(z)$, B(z) and $\bar{B}(z)$ are the N-1 th order polynomials, then we can write the Nth order forward filter as

$$F(z) + z^{N}\, B(z^{-1})\, C_N$$

and the Nth order, negative reflection coefficient, forward filter as

$$\bar{F}(z) - z^{N}\bar{B}(z^{-1})\, C_N \quad .$$

Substituting into the right side of the Nth order equation (III-44), we have

$$[\, F^{\dagger}(z^{-1}) + z^{-N}\, C_N^{\dagger}\, B^{\dagger}(z) \,]\, R(0)\, [\, \bar{F}(z) - z^{N}\bar{B}(z^{-1})\, C_N \,] \;+$$

$$[\, \bar{F}^{\dagger}(z^{-1}) - z^{-N}\, C_N^{\dagger}\, \bar{B}^{\dagger}(z) \,]\, R(0)\, [\, F(z) + z^{N}\, B(z^{-1})\, C_N \,] \;=$$

$$F^{\dagger}(z^{-1})\, R(0)\, \bar{F}(z) + \bar{F}^{\dagger}(z^{-1})\, R(0)\, F(z)$$

$$- C_N^{\dagger}[\, B^{\dagger}(z)\, R(0)\, \bar{B}(z^{-1}) + \bar{B}^{\dagger}(z)\, R(0)\, B(z^{-1}) \,]\, C_N$$

$$- z^{N}[\, F^{\dagger}(z^{-1})\, R(0)\, \bar{B}(z^{-1}) - \bar{F}^{\dagger}(z^{-1})\, R(0)\, B(z^{-1}) \,]\, C_N$$

$$+ z^{-N}\, C_N^{\dagger}[\, B^{\dagger}(z)\, R(0)\, \bar{F}(z) - \bar{B}^{\dagger}(z)\, R(0)\, F(z) \,] \quad . \qquad \text{(III-45)}$$

The last two terms in (III-45) are zero. To prove this, we note that the backward prediction error expression corresponding to (III-35) is

$$H^{\dagger}(z^{-1}) = \frac{R(0)}{2}\, \bar{B}(z^{-1})\, B^{-1}(z^{-1}) \text{ , or}$$

$$H(z) = \frac{1}{2}\, B^{\dagger -1}(z)\, \bar{B}^{\dagger}(z)\, R(0) \quad . \qquad \text{(III-46)}$$

This expression can be derived in the same manner as was (III-35) or by looking at (III-37). Using (III-35), we see that $B^{\dagger}(z)\, R(0)\, \bar{F}(z) = 2\, B^{\dagger}(z)\, H(z)\, F(z)$ and using (III-46), we see that $\bar{B}^{\dagger}(z)\, R(0)\, F(z) = 2\, B^{\dagger}(z)\, H(z)\, F(z)$ so the last term in (III-45) is zero. Likewise, for the next to last term in (III-45).

The N-1th order backward prediction error filter equation corresponding to (III-44) is

$$2 \, P'_{N-1} = [\, B^{\dagger}(z) \, R(0) \, \bar{B}(z^{-1}) + \bar{B}^{\dagger}(z) \, R(0) \, B(z^{-1}) \,] \, .$$

From this and N-1th order form of (III-44), we can rewrite (III-45) as

$$2 \, [\, P_{N-1} - C_N^{\dagger} \, P'_{N-1} \, C_N \,] \quad ,$$

which, from (III-15) and (III-17), is equal to $2 \, P_N$.