II-D. THE IMPORTANCE OF THE ($R(0)$, C_1, C_2 ...) DESCRIPTION

In previous sections of II, we have seen that the ($R(0)$, C_1, C_2, ...) description of the second order statistics of a stationary time series is fully equivalent to the more common autocorrelation function and power spectrum representations. In fact, the reflection coefficient description has very practical advantages over the other two when the second order statistics must be estimated from many short data samples.

In this section, we shall continue to elevate the importance of the ($R(0)$, C_1, C_2, ...) representation by showing how it can be used to create a new and perhaps better definition of a power spectrum. Following this, a more detailed study of the reflection coefficient sequence gives results which have both theoretical and practical importance.

1. A New Definition for Power Spectra

Following the definition of a power spectrum given by Wiener for a continuous time stationary process, the power spectrum of a band-limited stationary time series can be defined as

$$P(f) = \lim_{T\to\infty} P_T(f) ,$$

$$P_T(f) = \sum_{\tau=-T}^{T} \frac{|T+1-\tau|}{T+1} R(\tau) \cos 2\pi f \tau \Delta t , \qquad \text{(II-65)}$$

where $R(\tau)$ is the autocorrelation function at lags of $\tau\,\Delta t$ and is defined as

$$R(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n^* \, x_{n+\tau} \quad .$$

Here, x_n is the sample value of the time series at $n\Delta t$, where the bandwidth of the process is $W = 1/(2\Delta t)$. We note that

$$\frac{|T+1-\tau|}{T+1} \; R(\tau)$$

is the same as using a Bartlett weighting function on the autocorrelation function and thus $P_T(f) \geq 0$. Thus, $P(f)$ is defined as the limit of an infinite sequence of non-negative functions of f .

Instead of using (II-65), let us define $P_T(f)$ to be the maximum entropy spectrum corresponding to $R(\tau)$, $-T \leq \tau \leq T$. Then we again let $P(f)$ be the limit of this infinite sequence of non-negative functions. It is clear that under appropriate convergence criteria, this new definition using maximum entropy spectra will have the same limiting function, $P(f)$, as the old definition using the Bartlett window.

This new definition has two distinct advantages over the old definition, the first of which is that the sequence of maximum entropy spectra converges "faster" than the sequence of Bartlett spectra. Without attempting to define "faster" in a precise way, we should note that if the T th order maximum entropy spectrum is convolved with the T th order Bartlett window function, then we get the T th order Bartlett spectrum. This is easily seen to be true if we remember

(1) the autocorrelation function of the T th order maximum entropy spectrum agrees exactly with $R(\tau)$ for $-T \leq \tau \leq T$ and

(2) the Bartlett window multiplies an autocorrelation function by $|T+1-\tau|/(T+1)$ for $-T \leq \tau \leq T$ and by zero for $|\tau| > T$.

Thus, both the true autocorrelation function and the maximum entropy autocorrelation function will be converted into the Bartlett autocorrelation function by application of the Bartlett window. Therefore, as seen through the Bartlett window, the maximum entropy spectrum and the true spectrum look the same.

Actually, this is just a special case of how spectra generated from a variational principle for which (1) is true dominate spectra conventionally generated[11] from window functions as in (2). Any of the conventionally generated spectra can be obtained by applying its window function to any of the variationally generated spectra. We can now see that our "faster" convergence statement is based on the fact that the sequence of Bartlett spectra is a smoothed out version of the sequence of maximum entropy spectra.

The second advantage of the new definition is that it is simply related to the sequence of reflection coefficients. Since the T th order maximum entropy spectrum corresponds to the sequence $(R(0), C_1, C_2, \ldots, C_T, 0, 0, \ldots)$, the sequence of spectra simply corresponds to using more and more of the true values of the reflection coefficient sequence. We shall now look at the reflection coefficients as an infinite sequence of numbers and compare the properties of the sequence with that of the corresponding power spectrum.

2. <u>Spectral Properties in Terms of the ($R(0), C_1, C_2, \ldots$) Description</u>

In specifying the second order statistics of a stationary time series in terms of the ($R(0), C_1, C_2, \ldots$) description, we see that the zero lag value of the autocorrelation function is the scale factor for the spectrum and that the reflection coefficient sequence determines the shape of the spectrum. Most of our study in this section will be concerned with relating properties of the infinite sequence of reflection coefficients to those of the spectrum.

We already know that if $C_N = 1$, then the sequence terminates at C_N and our spectrum consists of a pure set of N delta functions. On the other hand, if a sequence becomes identically zero after C_N, then the spectrum is a N th order maximum entropy spectrum. To help our investigation of more general cases, we shall first see how the C's can be used to determine upper and lower bounds on the maximum entropy spectrum.

a. Upper and Lower Bounds on the Maximum Entropy Spectrum

From the Levinson algorithn, we see that if $H_{N-1}(f)$ is the fourier transform of the $N-1$ th order prediction error filter, then

$$H_N(f) = H_{N-1}(f) + C_N e^{-i2\pi Nf\Delta t} H^*_{N-1}(f) . \tag{II-66}$$

This equation is the frequency domain form of the Levinson algorithm and can be useful for calculation of spectra in special cases. By taking absolute values of (II-66), we find that

$$|H_{N-1}(f)| (1 - |C_N|) \le |H_N(f)| \le |H_{N-1}(f)| (1 + |C_N|) . \tag{II-67}$$

One might notice that there is at least one value of f at which the upper bound is reached and likewise for the lower bound. This is proven by the fact that the net phase shifts in going from $-W$ to $+W$ of $H_{N-1}(f)$ and $e^{-i2\pi Nf\Delta t} H^*_{N-1}(f)$ differ by 2π .

Starting with $H_0(f) = 1$, we see that

$$\prod_{n=1}^{N} (1 - |C_n|) \leq |H_N(f)| \leq \prod_{n=1}^{N} (1 + |C_n|) \quad . \tag{II-68}$$

For $N > 1$, it is not necessarily true that either of the bounds in (II-68) will be achieved. However the bounds in (II-68) are tight bounds in the sense that one of them can be reached at any given frequency by properly adjusting the phases of the reflection coefficients. Now, since

$$P_N = R(0) \prod_{n=1}^{N} (1 - |C_n|^2) \; ,$$

we can write

$$\frac{R(0) \prod_{n=1}^{N} (1 - |C_n|^2)}{2W \prod_{n=1}^{N} (1 + |C_n|)^2} \leq P_N(f) \leq \frac{R(0) \prod_{n=1}^{N} (1 - |C_n|^2)}{2W \prod_{n=1}^{N} (1 - |C_n|)^2} \; , \text{ or}$$

$$\prod_{n=1}^{N} \frac{1 - |C_n|}{1 + |C_n|} \leq \frac{P_N(f)}{R(0)/2W} \leq \prod_{n=1}^{N} \frac{1 + |C_n|}{1 - |C_n|} \quad . \tag{II-69}$$

If $Q_N = \prod_{n=1}^{N} \frac{1 + |C_n|}{1 - |C_n|}$, then we have

$$- \log Q_N \leq \log P_N(f) - \log[R(0)/2W] \leq \log Q_N \quad . \tag{II-70}$$

Since R(0) / 2W is the average value of the spectral density, we see that if our spectrum is plotted on a db scale (which is almost always the proper thing to do), then our limits are equally spaced $10 \log Q_N$ above and below the db value of the average spectral density.

b. The Average Value of the Logarithm of the Spectrum

We have previously stated and used the fact that

$$\frac{1}{2W} \int_{-W}^{+W} \ln P(f)\, df = \ln(P_\infty/2W) . \qquad \text{(II-71)}$$

We shall now prove this by first showing that if F(f) is a minimum phase filter whose first weight is unity, i.e., an optimum least mean square prediction error filter for a stationary time series which is not perfectly predictable, then

$$\int_{-W}^{+W} \ln F(f) \, df = 0 . \qquad \text{(II-72)}$$

With $z = e^{-i2\pi f\Delta t}$ and $df = dz / (-i2\pi\Delta t\, z)$, we can express F(f) as $1 + a_1 z + a_2 z^2 + \ldots$, where this z transform is analytic and has no zeros on or inside the unit circle. Thus (II-72) becomes

$$\frac{1}{i2\pi\Delta t} \oint \ln (1 + a_1 z + a_2 z^2 + \ldots) \, z^{-1} \, dz \qquad \text{(II-73)}$$

where the integration is around the unit circle in the counterclockwise direction.

From Cauchy's integral formula, we have that

$$g(0) = \frac{1}{2\pi i} \oint \frac{g(z)}{z} \, dz$$

if $g(z)$ is analytic on and inside the contour of integration, where the contour encloses the origin. Because of the minimum phase condition, $\ln(1 + a_1 z + a_2 z^2 + \ldots)$ is an analytic function on and inside the unit circle and thus (II-73) is equal to

$$\frac{1}{\Delta t} \ln (1 + a_1 z + a_2 z^2 + \ldots) \Big|_{z=0} = 0 \ .$$

Equation (II-72) is also true if $F(f)$ is maximum phase and has unity for the z^0 coefficient. In this case we can write $F(f) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots$, where the z transform has no zeros and is analytic on and outside the unit circle. By letting $y = z^{-1}$ so that $y^{-1} dy = - z^{-1} dz$, we have

$$\oint \ln (1 + a_1 z^{-1} + a_2 z^{-2} + \ldots) z^{-1} dz =$$

$$- \oint \ln (1 + a_1 y + a_2 y^2 + \ldots) y^{-1} dy = 0 ,$$

since $1 + a_1 y + a_2 y^2 + \ldots$ is minimum phase.

We now see that if a time series can be generated by passing white noise of variance P_∞ through a minimum phase filter $F(z)$ so that its spectrum $P(f)$ is $|F(z)|^2 P_\infty/2W$, then

$$\int_{-W}^{+W} \ln P(f) \, df = \int_{-W}^{+W} \ln(P_\infty/2W) + \int_{-W}^{+W} \ln F^*(z^{-1}) \, df$$

$$+ \int_{-W}^{+W} \ln F(z) \, df = \int_{-W}^{+W} \ln(P_\infty/2W) \, df = 2W \ln(P_\infty/2W) \ .$$

Thus the average value of the logarithm of the spectrum is $\ln(P_\infty/2W)$. Since

$$P_\infty = \frac{R(0)}{2W} \prod_{n=1}^{\infty} (1 - |C_n|^2) \quad , \text{ we have}$$

$$\frac{1}{2W} \int_{-W}^{+W} \ln P(f) \, df = \ln(R(0)/2W) + \sum_{n=1}^{\infty} \ln(1 - |C_n|^2) \, . \qquad \text{(II-74)}$$

c. Limit Properties of the Reflection Coefficient Sequence

One of the major properties of the reflection coefficient sequence is that it converges to zero if the mean square error of the infinite prediction error filter does not vanish. That is, if $P_\infty > 0$, then $C_n \to 0$. We prove this by noting that

$$\ln (1 - |C_n|^2) \le - |C_n|^2 \quad \text{for} \quad |C_n| < 1 \, .$$

Thus $\ln P_\infty = \ln R(0) + \sum_{n=1}^{\infty} \ln(1 - |C_n|^2) \le \ln R(0) - \sum_{n=1}^{\infty} |C_n|^2$.

Therefore, if $P_\infty > 0$, $\sum_{n=1}^{\infty} |C_n|^2$ must converge and thus $C_n \to 0$.

An equivalent statement is that if $C_n \not\to 0$, then $P_\infty = 0$. However, $C_n \to 0$ does not mean that $P_\infty > 0$. An example of this is $C_1 = 0$, $C_n = 1/\sqrt{n}$ for $n=2$ to ∞ . Then

$$\ln (1 - |C_n|^2) = \ln (1 - \frac{1}{n}) < - \frac{1}{n} \quad , \text{ and}$$

$$\ln P_\infty \le \ln R(0) - \sum_{n=2}^{\infty} \frac{1}{n} = - \infty \, .$$

Thus $\ln P_\infty = - \infty$ or $P_\infty = 0$.

From (II-69) and (II-70), we see that if Q_∞ is finite, then $P_\infty > 0$ since the spectrum will be bounded above zero and thus $\int \ln P(f) \, df$ will be finite. However, Q_∞ can be infinite with P_∞ still being finite. An example is white noise plus a delta function.

Without giving any proofs, if a power spectrum is identically zero over some frequency range, then $P_\infty = 0$ but $C_n \nrightarrow 0$. A power spectrum for which $P_\infty = 0$ but $C_n \to 0$ is $P(f) = \exp(-1/|f|), -W \le |f| \le W$. If a delta function is added to this spectrum at $f=0$, then we have an example of a spectrum which is positive everywhere but whose time series is perfectly predictable.

3. Plotting Maximum Entropy Spectra

The most common error made by a new user of maximum entropy spectra is not calculating and plotting the spectral function on a fine enough frequency grid. This error is often induced by familiarity with the fast fourier transform which calculates values on a dense enough set of frequencies so that the transform is reversible. It is overlooked that the maximum entropy spectrum is proportional to the reciprocal of the power response of a filter and thus can change its magnitude very quickly. For spectra with sharp peaks, a coarse frequency grid will give a highly misleading plot of the spectrum. This has caused users to report that the maximum entropy spectrum is erratic and inaccurate with respect to the location and power of different spectral peaks. Thus, one good question is how densely should the spectra be plotted. Another question that always occurs when a graph is to be made is where to center the plot and what is its dynamic range. These questions can be answered in terms of the numbers $R(0)$, C_1, C_2, ..., C_N which are known before the fourier transform is calculated.

With respect to centering the graph, one knows that the average value of the spectrum is given by $R(0)/2W$ and that the average value of the logarithm of the spectrum is given by $\log(P_\infty/2W)$. Finally, since (II-70) gives upper and lower bounds for the spectrum in terms of Q_N , we see that the dynamic range problem is easily solved.

To form a conservative estimate of how densely the spectrum should be plotted in frequency, let us assume that the N th order spectrum has N separate peaks, all of which have the same bandwidth and achieve the upper bound of $R(0)\, Q_N / 2W$. Assuming that all of the power is in these peaks, their bandwidths can be estimated to be $2W/NQ_N$. If we wish to have at least two points per this bandwidth, then we need at least $2NQ_N$ points from $-W$ to W, or $\Delta f = W/NQ_N$. Of course, for small values of N, this spacing would be too coarse. However, for N greater than twenty, this spacing will probably be reasonable.

In using maximum entropy spectra, it should be remembered that they are spectral density estimates. Thus, the amount of power in a small bandwidth is not the peak value of the spectral density function, but is its integral over the given bandwidth. Following this thought, it should be stated that the peak value and the bandwidth of a spectral line strongly depend on the level of the "background" spectrum. Thus, one can expect these quantities to have considerable variance in maximum entropy estimates from real data. However, their product, which is proportional to the total power in the spectral line, will be estimated quite accurately.

It is difficult to visually estimate the power in a peak from a db spectral plot. Because of this, a plot of the integrated power spectra, i.e.,

$$\int_{-W}^{f} P(f)\, df ,$$

can be very useful. This function, which should be plotted on a linear vertical scale (not in db), goes from zero to $R(0)$. Of course, if one has not sampled the spectrum densely enough, the numerical integration may not be very close to $R(0)$. This is a powerful clue that the spectrum as plotted is not a good representation of the true estimated spectrum and that a denser set of points in frequency is needed, especially at the peaks in the spectrum.

Appendix A gives an analytic expression for the integrated power spectrum, although the complexity of the equation makes its calculation impractical. However, if one has a sharp peak in the spectrum and the sampling in frequency was not dense enough to accurately plot the peak, then a simple curve fitting technique can be used. If there are three points on the spectral peak which are well above the background level, then the functional form

$$\frac{A}{1 + \frac{(f - f_0)^2}{(B.W./2)^2}}$$

can be fitted to the three points. We might note that this function corresponds to a single pole analog filter, where A = peak value, f_0 = center frequency and B.W. = bandwidth ($\pm$ 3 db down points). An estimate of the total power in the peak is obtained by noting that

$$\int_{-\infty}^{+\infty} \frac{A}{1 + \frac{(f - f_0)^2}{(B.W./2)^2}} \, df = \frac{\pi}{2} A \cdot (B.W.) .$$

There are some observations to be made concerning numerical accuracy in calculating maximum entropy spectra. It should be noticed that when there is a sharp, high peak in the spectrum, this means that the fourier transform of the prediction error filter is very small in magnitude at the peak frequency. Thus in the fourier transform calculation, we are summing together a set of N numbers which almost cancel themselves out. This is of course bad from a numerical accuracy point of view. One way of deciding if the number of decimal places in the arithmetic is sufficient is to note that an estimate of the possible dynamic range of the spectrum is

$$Q_N^2 = \prod_{n=1}^{N} \left(\frac{1 + |C_n|}{1 - |C_n|} \right)^2 ,$$

which is the ratio of the upper bound to the lower bound of the spectrum. Thus, the accuracy of the calculation of peak values in the spectrum will be governed by the numerical precision in excess of Q_N^2 .