II-B. EXTENDING THE AUTOCORRELATION FUNCTION

The general problem of starting with an incomplete set of autocorrelation function lag values and determining some particular power spectrum that is consistent with these values is equivalent to determining a particular completion of the infinite autocorrelation function. In this section, we shall look at the practical problem in which the known autocorrelation lag values are $R(n)$, $|n| \leq N$. Thus, specifying a consistent power spectrum is identical to specifying an infinite, allowable, extension of these first $N+1$ lag values. By allowable, we mean that the fourier transform of the extended function is non-negative, i.e., a power spectrum.

It should be clearly recognized that this section deals with the general problem of spectral estimation from autocorrelation information. In this context, we shall develop and prove the Fundamental Autocorrelation Matrix Theorem which governs the general extension of autocorrelation functions. Later on, we shall see that the special extension which corresponds to generating a maximum entropy spectrum has some particularly appealing properties.

In section II-A, the solution to equation (II-15) was derived and shown to be unique under the assumption that the N by N Toeplitz submatrix was positive definite. The following useful theorem generalizes this result.

1. The Shortest Prediction Error Filter Theorem

A solution to the Nth order prediction error filter equation

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} \begin{Bmatrix} 1 \\ a_1 \\ \\ a_N \end{Bmatrix} = \begin{Bmatrix} P_N \\ 0 \\ \\ 0 \end{Bmatrix} \qquad \text{(II-22)}$$

always exists if the N+1 by N+1 hermitian Toeplitz matrix is non-negative definite. If the rank of the matrix is less than N, then there is more than one solution to the equation. However, the solution corresponding to the shortest prediction error filter is unique.

Proof: The cases inwhich the rank of the matrix is either zero or N+1 are trivial. Thus, suppose the rank is M where $N \geq M \geq 1$. Then the solution to the M+1 by M+1 equation is unique and of the form

$$\begin{bmatrix} R(0) & R(-1) & \cdots & R(-M) \\ R(1) & R(0) & \cdots & R(1-M) \\ & & & \\ R(M) & R(M-1) & \cdots & R(0) \end{bmatrix} \begin{Bmatrix} 1 \\ b_1 \\ \\ b_M \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \\ 0 \end{Bmatrix}, \qquad \text{(II-23)}$$

where b_M is not zero. Using (II-23), we can write for the next higher order matrix,

$$\begin{bmatrix} R(0) & R(-1) & R(-M) & R(-1-M) \\ R(1) & R(0) & R(1-M) & R(-M) \\ & & & \\ R(M) & R(M-1) & R(0) & R(-1) \\ R(M-1) & R(M) & R(1) & R(0) \end{bmatrix} \begin{Bmatrix} 1 \\ b_1 \\ \\ b_M \\ 0 \end{Bmatrix} + q \begin{Bmatrix} 0 \\ b_M^* \\ \\ b_1^* \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \\ 0 \\ \Delta \end{Bmatrix} + q \begin{Bmatrix} \Delta^* \\ 0 \\ \\ 0 \\ 0 \end{Bmatrix}$$

where Δ is defined implicitly and q is an arbitrary complex number. If we now premultiply by the complex conjugate transpose of the left-side column vector, we end up with the real scalar $q\Delta^* + q^*\Delta$. If Δ is not zero, then q can be chosen to make this quantity negative which contradicts the non-negative definiteness condition. Thus, Δ must be zero. Thus, choosing q to be zero, which is equivalent to augmenting the Mth order prediction error filter by a zero, we have the shortest solution to the M+1 th order equation. Continuing with this argument, we see that the Mth order filter is also a solution to the Nth order

equation. Also, it is clearly both the shortest such filter and is unique.

The next theorem concerns a Cholesky type factorization of the Toeplitz matrix. The usual Cholesky factorization requires that the matrix be positive definite. This condition is not necessary for the Toeplitz case.

2. Factorization of the Toeplitz Matrix

Given an N+1 by N+1 non-negative definite hermitian Toeplitz matrix, one can uniquely solve the matrix equation

$$\begin{bmatrix} R(0) & R(-1) & R(-2) & R(-N) \\ R(1) & R(0) & R(-1) & R(1-N) \\ R(2) & R(1) & R(0) & R(2-N) \\ R(N) & R(N-1) & R(N-2) & R(0) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & b_1 & 1 & 0 \\ a_N & b_{N-1} & c_{N-2} & 1 \end{bmatrix} = \begin{bmatrix} P_N & * & * & * \\ 0 & P_{N-1} & * & * \\ 0 & 0 & P_{N-2} & * \\ 0 & 0 & 0 & P_0 \end{bmatrix} \tag{II-24}$$

where the * terms are not required explicitly. This equation is in effect (II-22) N+1 times. We make (II-24) unique when the Toeplitz matrix is singular by using the shortest prediction error filters. Premultiplying (II-24) by the complex conjugate transpose of the prediction error filter matrix we have

$$\begin{bmatrix} 1 & a_1^* & a_N^* \\ 0 & 1 & b_{N-1}^* \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R(0) & R(-1) & R(-N) \\ R(1) & R(0) & R(1-N) \\ R(n) & R(N-1) & R(0) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_N & b_{N-1} & 1 \end{bmatrix} = \begin{bmatrix} P_N & 0 & 0 \\ 0 & P_{N-1} & 0 \\ 0 & 0 & P_0 \end{bmatrix}. \tag{II-25}$$

The right-hand matrix is diagonal because it was obtained from (II-24) by multiplying two upper triangular matrices together and thus must be upper triangular. However, the left-hand side of (II-25) is also

hermitian and thus must be diagonal. Now, since the prediction error filter matrix is clearly non-singular, one can express the autocorrelation matrix in the Cholesky type factorization of

$$\begin{bmatrix} R(0) & R(-1) & R(-N) \\ R(1) & R(0) & R(1-N) \\ R(N) & R(N-1) & R(0) \end{bmatrix} = \begin{bmatrix} 1 & a_1^* & a_N^* \\ 0 & 1 & b_{N-1}^* \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} P_N & 0 & 0 \\ 0 & P_{N-1} & 0 \\ 0 & 0 & P_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_N & b_{N-1} & 1 \end{bmatrix}^{-1} . \tag{II-26}$$

One important consequence of this theorem is that any non-negative definite Toeplitz matrix can be extended indefinitely to higher order non-negative definite Toeplitz matrices. That is, if the Nth order Toeplitz matrix is non-negative definite, then there exists an N+1 th order non-negative definite Toeplitz matrix with the Nth order matrix as a submatrix. To prove this, we note that there is always a unique value for R(N+1) such that

$$\begin{bmatrix} R(0) & R(-1) & R(-N-1) \\ R(1) & R(0) & R(-N) \\ R(N) & R(N-1) & R(-1) \\ R(N+1) & R(N) & R(0) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_N & a_{N-1} & 0 \\ 0 & a_N & 1 \end{bmatrix} = \begin{bmatrix} P_N & * & * \\ 0 & P_N & * \\ 0 & 0 & * \\ 0 & 0 & P_0 \end{bmatrix} .$$

One can then express this N+1 th order matrix in the form of (II-26). Since the P's are non-negative, the N+1 th order matrix will be non-negative definite. It is clear that this extension of the matrix can be continued indefinitely.

The study of sampled autocorrelation functions relies almost completely on the following theorem.

3. The Fundamental Autocorrelation Matrix Theorem

Given a set of complex numbers, $R(n)$, $|n| \leq N$, where $R(n) = R^*(-n)$, then these numbers are the beginning of an autocorrelation function if and only if the matrix

$$\begin{bmatrix} R(0) & R(-1) & & R(-N) \\ R(1) & R(0) & & R(1-N) \\ & & & \\ R(N) & R(N-1) & & R(0) \end{bmatrix} \qquad \text{(II-27)}$$

is non-negative definite.

To prove the necessary condition of this theorem, we begin with the definition of the autocorrelation function of the stationary time series, x_s , as

$$R(n) = \overline{x_s^* \, x_{s+n}} \quad . \qquad \text{(II-28)}$$

Knowing $R(n)$, we can write the average square value of $y_n = \sum_{s=0}^{N} a_s \, x_{n-s}$, where the a's are arbitrary, as

$$\overline{y_n^* y_n} = \overline{\sum_{s=0}^{N} a_s^* x_{n-s}^* \sum_{r=0}^{N} a_r x_{n-r}} =$$

$$= \sum_{s=0}^{N}\sum_{r=0}^{N} a_s^* a_r \overline{x_{n-s}^* x_{n-r}} = \sum_{s=0}^{N}\sum_{r=0}^{N} a_s^* a_r R(s-r) =$$

$$\{a_0^* a_1^* \dots a_N^*\} \begin{bmatrix} R(0) & R(-1) & \dots & R(-N) \\ R(1) & R(0) & \dots & R(1-N) \\ & & & \\ R(N) & R(N-1) & \dots & R(0) \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ \\ a_N \end{Bmatrix} . \qquad \text{(II-29)}$$

Since $\overline{y_n^* y_n} \geq 0$ and the a's are arbitrary, we see that the matrix (II-27) must be non-negative definite. Thus, we have proved that if the $R(n)$, $|n| \leq N$, are the beginning of an autocorrelation function, then the N+1 by N+1 Toeplitz matrix formed from these values is necessarily non-negative definite.

To prove the sufficiency part of the theorem, we shall use a procedure similar to the one that Wiener used in the continuous case. The proof uses the argument that the average square output of any narrow-band filter must be non-negative.

The digital filter

$$\frac{1}{\sqrt{N}} \left(1 + \alpha z + \alpha^2 z^2 + \dots + \alpha^{N-1} z^{N-1} \right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \alpha^n z^n = \frac{1}{\sqrt{N}} \frac{1-\alpha^N z^N}{1-\alpha z} ,$$

where $\alpha = e^{i2\pi f_0 \Delta t}$, has the power response of

$$B_N(f,f_0) = \frac{1}{N} \frac{1-\alpha^N z^N}{1-\alpha z} \frac{1-\alpha^{-N} z^{-N}}{1-\alpha^{-1} z^{-1}} = \frac{\sin^2[\pi N \Delta t (f-f_0)]}{N \sin^2[\pi \Delta t (f-f_0)]} . \qquad \text{(II-30)}$$

The integral of $B_N(f,f_0)$ from $-W$ to $+W$ is unity. Furthermore, if f does not lie within Δf of f_0 (taking aliasing into consideration), the maximum value of $B_N(f,f_0)$ is less than $1/N(\pi\Delta t\Delta f)^2$. Thus, as $N \to \infty$, the entire unity area of $B_N(f,f_0)$ begins to lie within an arbitrarily small Δf of f_0.

Let us assume the converse of the fundamental theorem, namely, that for all functions $Q(f)$, such that

$$Q(f) = \frac{1}{2W} \sum_{n=-\infty}^{+\infty} R(n)\, e^{-i2\pi fn\Delta t} ,$$

where for $|n| \leq N$ the $R(n)$ are the given $R(n)$, $Q(f)$ will be negative at some frequency, f_1. This means that $Q(f)$ must be negative within some region Δf about f_1. Thus, for N large enough, say $N=M$, we will have

$$\int_{-W}^{+W} B_M(f,f_1)\, Q(f)\, df < 0 . \qquad \text{(II-31)}$$

But this integral is also equal to

$$\frac{1}{M} \{ 1 \quad \alpha_1^{-1} \quad \alpha_1^{-2} \quad \cdots \quad \alpha_1^{-M+1} \} \begin{bmatrix} R(0) & R(-1) & \cdots & R(-M+1) \\ R(1) & R(0) & \cdots & R(-M+2) \\ \vdots & \vdots & & \vdots \\ R(M-1) & R(M-2) & \cdots & R(0) \end{bmatrix} \begin{Bmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_1^{M-1} \end{Bmatrix} ,$$

where $\alpha_1 = e^{i2\pi f_1 \Delta t}$. Thus, the M by M matrix is not non-negative definite. Therefore, assuming the converse of the fundamental theorem, we conclude that there are some non-negative definite Toeplitz matrices which cannot be extended indefinitely in a non-negative definite way. Since the factorization theorem has shown this to be false, we have proved the sufficiency part of the fundamental autocorrelation matrix theorem.

4. Allowable Values for the Next Autocorrelation Lag

Suppose we have the first N values of an autocorrelation function, R(n) , (n=0 to N-1), and we wish to determine what values are possible for R(N) . If we form the N+1 by N+1 autocorrelation matrix, the Fundamental Autocorrelation Matrix Theorem tells us that the N by N Toeplitz submatrix will be non-negative definite and that R(N) will have a permissible value only if the full N+1 by N+1 matrix is also non-negative definite.

Let us first look at the situation in which the N by N Toeplitz submatrix is singular. In this case, if (1, b_1, ..., b_M) is the shortest prediction error filter, we see from (II-23) and the argument following that equation, that for the N+1 by N+1 matrix to be non-negative definite, we must have

$$\sum_{n=0}^{M} R(N-n)\, b_n = 0 \ , \ (b_0 = 1) \ , \text{ or}$$

$$R(N) = - \sum_{n=1}^{M} R(N-n)\, b_n \ . \qquad \text{(II-32)}$$

Thus, if an autocorrelation matrix is singular, the next value of the autocorrelation function is uniquely specified. It then follows by simple induction that the entire autocorrelation function is determined. We shall see that this situation means that the autocorrelation function consists of a finite set of cosine functions, i.e., the spectrum consists of a set of delta functions and the time function is a sum of pure frequencies.

If the N by N Toeplitz submatrix is non-singular, then the only condition on the N+1 by N+1 matrix is that its determinant be non-negative. In discussing (II-15),we noticed that the determinant of the N+1 by N+1 matrix was equal to P_N times the determinant of the N by N sub-matrix. Thus our requirement is that P_N be non-negative, which from (II-20), means that $|c_N| \leq 1$. By combining (II-17) and (II-18), we can write

$$R(N) = - \sum_{n=1}^{N-1} R(N-n)\, b_n - c_N\, P_{N-1} \quad . \qquad \text{(II-33)}$$

If $|c_N| \leq 1$, we see that the value of R(N) in the complex plane lies on or inside a circle of radius P_{N-1} centered at $- \sum_{n=1}^{N-1} R(N-n)\, b_n$.

If one chooses a value for R(N) and calculates the next higher order prediction error filter and its mean square error, P_N , then (II-33) can be used again to determine the permissible values for R(N+1) . In this manner, one can generate any permissible extension of the autocorrelation function. It is interesting to note that since $P_X = P_{X-1}(1 - |c_X|^2)$, the radii of the sequence of circles cannot increase and will decrease unless the c's are zero. That is, if one chooses a value for R(X) other than the center of its circular range, the circle for R(X+1) will have a reduced radius. In fact,

if $R(X)$ is chosen to be on the perimeter of its circular range, i.e., $|c_X| = 1$, P_X will be zero and there is no longer any choice in selecting the value of $R(X+1)$. This moves (II-33) into the singular situation that is governed by (II-32).

From the above observations, if one had to make the "best" selection for $R(X)$, choosing the value in the center of the circle is the most appealing. One "argument" for the center is that is is the only unique point in the disk. Any other point has an infinite number of other points of equal radius, etc., and to argue that a particular one of them is better than the rest would be difficult. A second "reason" for choosing the center is that this gives one the biggest disk for selecting the value of $R(X+1)$. That is, one maintains the maximum freedom of choice in the further extension of the autocorrelation function. A third "reason" could be that selecting a value for $R(X)$ on the perimeter should be a "worst" choice since a spectrum consisting of a pure set of spectral lines is improbable. The center of the disk is the point that is the furthest from these bad points. As will be shown later, if one extends the autocorrelation function to infinity by always selecting the center value for each consecutive lag value, one generates the autocorrelation function corresponding to the maximum entropy spectrum.

We are now ready to prove the following theorem which relates the c_n of (II-18) to the autocorrelation function and vice versa.

5. The Autocorrelation-Reflection Coefficient Theorem

There exists a one to one correspondence between an autocorrelation function and the set of numbers, $(R(0) , c_1 , c_2 , \ldots)$, where $R(0)$ is the zero lag value of the autocorrelation function and is thus real and non-negative and where the c_n are the reflection

coefficients of (II-18) with the $|c_n| \leq 1$. This set of numbers terminates with $R(0)$ if $R(0) = 0$ or at c_n if $|c_n| = 1$.

Proof: If one is given the set $(R(0) , c_1 , c_2 , \ldots)$, then one can recursively generate the unique set of numbers $(R(0), R(1), R(2), \ldots)$ by using (II-33). This is seen to be true if one notes that the b_n and P_{N-1} values needed to find $R(N)$ are uniquely determined by $(R(0), R(1), \ldots, R(N-1))$. If $|c| = 1$ at some stage, then the mean square error will become zero and the further extension of the autocorrelation function will be governed by (II-32). Thus, the sequence of reflection coefficients can be terminated.

Likewise, if one is given the set $(R(0), R(1), R(2), \ldots)$, then the unique set $(R(0), c_1, c_2, \ldots)$ can be generated by inversion of (II-33) as long as the P_{N-1} is non-zero. Of course, if $P_{N-1} = 0$, then $|c_{N-1}|$ must have been equal to one and the c sequence can be terminated since the values of $R(N), R(N+1), \ldots$ are then rigidly determined by the condition that the input set $(R(0), R(1), R(2), \ldots)$ be an autocorrelation function. A simple corollary of this theorem is that there is a one-to-one correspondence of the finite sets $(R(0), R(1), R(2), \ldots, R(N))$ and $(R(0), c_1, c_2, \ldots, c_N)$ with a suitable statement made about the special case when some c has unit magnitude.

The most important observation to be made about the Autocorrelation-Reflection Coefficient Theorem is that it shows that the set $(R(0), c_1, c_2, \ldots)$ gives us a new representation of the second order statistics of a stationary time series. The best known representations are, of course, the autocorrelation function and the power spectrum. The "best" representation normally depends on how and why the time series is being analyzed. It will be seen later in

chapter III that the reflection coefficient representation is particularly well suited for estimating the second order statistics from finite data samples of the stationary time series.

6. The Prediction Error Filter - Minimum Phase Theorem

Our proof of the sufficiency part of the Fundamental Autocorrelation Matrix Theorem was by contradiction. In the next section, we shall prove the sufficiency part of this theorem for positive definite autocorrelation matrices by actually constructing the corresponding maximum entropy spectrum. To do this first requires that we prove that the prediction error filters as generated by (II-19) are minimum phase, i.e., they have all of their zeros outside of the unit circle. The proof will use the following simplified theorem by Rouché.

If $F(z)$ is a polynomial in z and $G(z)$ is also a polynomial in z, but where $|G(z)| < |F(z)|$ on the unit circle, then $F(z)$ and $F(z) + G(z)$ have the same number of roots inside the unit circle.

The proof uses the fact that if a polynomial $H(z)$ has N zeros inside the unit circle, then as one moves z on a complete counter-clockwise circuit around the unit circle, the net phase shift in $H(z)$ is $2\pi N$, and vice versa. We write

$$F(z) + G(z) = F(z) \left[1 + G(z) / F(z) \right]$$

and note that the net phase shift in $F(z) + G(z)$ as z goes around the unit circle will be the sum of the net phase shifts in $F(z)$ and $1 + G(z) / F(z)$. Since $|G(z)| < |F(z)|$ on $|z| = 1$, the real part of $1 + G(z) / F(z)$ must always be positive and thus it can have

no net phase shift as z goes around the unit circle. Thus, $G(z) + F(z)$ and $F(z)$ have the same net phase shift and thus have the same number of zeros inside the unit circle.

We can now show that if the magnitudes of all of the reflection coefficients in the sequence $(c_1, c_2, \ldots, c_N)$ are less than unity, which means that the Nth order autocorrelation matrix is positive definite, then the Nth order prediction error filter (P.E.F.) is minimum phase.

Let $F(z) = \sum_{n=0}^{N-1} b_n z^n$ be the N-1th order prediction error filter, where $b_0=1$. Then if $G(z) = c_N \sum_{n=0}^{N-1} b_n^* z^{N-n} = c_N z^N F^*(z^{-1})$, where $|c_N| < 1$, then when $|z| = 1$, we have

$$|G(z)| < |z^N F^*(z^{-1})| = |z^N [F(z)]^*| = |F(z)| .$$

From (II-19), we see that the Nth order prediction error filter is $F(z) + G(z)$. Thus, if the N-1th order P.E.F. has no zeros inside the unit circle, neither does the Nth order P.E.F. This induction proof is completed by noting that the zero th order P.E.F. has no zeros at all and is thus minimum phase.

A different and perhaps more basic approach is used in III-C.2 to prove that multichannel prediction error filters are minimum phase. The multichannel proof of course applies to the single channel case.

7. The Maximum Entropy Extension

Suppose we have the first N+1 values of a positive definite autocorrelation function and we solve (II-15) for the Nth order prediction error filter, $A(z) = 1 + a_1 z + a_2 z^2 + \ldots + a_N z^N$. If we now choose the zero extension of the reflection coefficient sequence, starting with c_{N+1} , we see from (II-32) or (II-33) that the corresponding extension of the autocorrelation function will be given by the convolutional feedback operation of

$$R(m) = -\sum_{n=1}^{N} R(m-n)\, a_n \quad , \quad m > N \quad , \quad \text{or} \tag{II-34}$$

$$\sum_{n=0}^{N} R(m-n)\, a_n = 0 \quad , \quad m > N \quad , \quad a_0 = 1 \ .$$

Let us define the function $H(z)$ as

$$H(z) = R(0)/2 + R(1)z + R(2)z^2 + \ldots \quad . \tag{II-35}$$

It is clear that $H(z) + H^*(z^{-1})$ is the z transform of the autocorrelation function. Furthermore, since the cosine transform of $H(z)$ is one half of the cosine transform of the autocorrelation function, we see that the real part of $H(z)$ is non-negative. If we convolve $H(z)$ with the Nth order prediction error filter, $A(z)$, the result must be of the form

$$H(z)\, A(z) = \frac{R(0)}{2}\left[\, 1 + d_1 z + d_2 z^2 + \ldots + d_N z^N \,\right] = \frac{R(0)}{2}\, D(z) \ , \tag{II-36}$$

since (II-34) tells us that $d_m = 0$ for $m > N$. We see from (II-36) that

$$H(z) = \frac{R(0)}{2}\, \frac{D(z)}{A(z)} \tag{II-37}$$

and that $H(z)$ is a legitimate z-transform expression, that is, as a Taylor series in z, it converges on the unit circle because $A(z)$ has no zeros inside the unit circle.

We now wish to derive the relationship between $D(z)$ and $A(z)$. The derivation is by induction and will be illustrated by going from $N=2$ to $N=3$.

The first three terms in the convolution of the second order prediction error filter, $1 + a_1 z + a_2 z^2$, with $H(z)$ can be written in a matrix equation as

$$\begin{bmatrix} R(0)/2 & 0 & 0 \\ R(1) & R(0)/2 & 0 \\ R(2) & R(1) & R(0)/2 \end{bmatrix} \begin{Bmatrix} 1 \\ a_1 \\ a_2 \end{Bmatrix} = \frac{R(0)}{2} \begin{Bmatrix} 1 \\ d_1 \\ d_2 \end{Bmatrix} . \qquad \text{(II-38)}$$

This 2nd order prediction error filter of course obeys the Toeplitz matrix equation

$$\begin{bmatrix} R(0) & R(-1) & R(-2) \\ R(1) & R(0) & R(-1) \\ R(2) & R(1) & R(0) \end{bmatrix} \begin{Bmatrix} 1 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} P_2 \\ 0 \\ 0 \end{Bmatrix} . \qquad \text{(II-39)}$$

Subtracting (II-38) from (II-39), one gets

$$\begin{bmatrix} R(0)/2 & R(-1) & R(-2) \\ 0 & R(0)/2 & R(-1) \\ 0 & 0 & R(0)/2 \end{bmatrix} \begin{Bmatrix} 1 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} P_2 \\ 0 \\ 0 \end{Bmatrix} - \frac{R(0)}{2} \begin{Bmatrix} 1 \\ d_1 \\ d_2 \end{Bmatrix} .$$

Taking the complex conjugates of these equations and rearranging the order, one finds that

$$\begin{bmatrix} R(0)/2 & 0 & 0 \\ R(1) & R(0)/2 & 0 \\ R(2) & R(1) & R(0)/2 \end{bmatrix} \begin{Bmatrix} a_2^* \\ a_1^* \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P_2 \end{Bmatrix} - \frac{R(0)}{2} \begin{Bmatrix} d_2^* \\ d_1^* \\ 1 \end{Bmatrix} \qquad \text{(II-40)}$$

Using (II-38) and (II-40), we can generate the 3rd order equivalent of (II-38) by

$$\begin{bmatrix} R(0)/2 & 0 & 0 & 0 \\ R(1) & R(0)/2 & 0 & 0 \\ R(2) & R(1) & R(0)/2 & 0 \\ R(3) & R(2) & R(1) & R(0)/2 \end{bmatrix} \left\{ \begin{Bmatrix} 1 \\ a_1 \\ a_2 \\ 0 \end{Bmatrix} + c_3 \begin{Bmatrix} 0 \\ a_2^* \\ a_1^* \\ 1 \end{Bmatrix} \right\} =$$

$$\frac{R(0)}{2} \begin{Bmatrix} 1 \\ d_1 \\ d_2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \Delta_3 \end{Bmatrix} + c_3 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ P_2 \end{Bmatrix} - c_3 \frac{R(0)}{2} \begin{Bmatrix} 0 \\ d_2^* \\ d_1^* \\ 1 \end{Bmatrix} = \frac{R(0)}{2} \left\{ \begin{Bmatrix} 1 \\ d_1 \\ d_2 \\ 0 \end{Bmatrix} - c_3 \begin{Bmatrix} 0 \\ d_2^* \\ d_1^* \\ 1 \end{Bmatrix} \right\} . \qquad \text{(II-41)}$$

We note that the last equality is valid because $\Delta_3 + c_3 P_2 = 0$.

Applying inductive reasoning to (II-41), we see that the D(z) polynomials are built up from the sequence of reflection coefficients in exactly the same manner as are the prediction error filters, except that the negatives of the reflection coefficients are used. Since, if we have the Nth order prediction error filter, we can uniquely decompose it into its sequence of N reflection coefficients, we can then build up the corresponding Nth order D(z) polynomial by using the negative of the reflection coefficient sequence. We also go from D(z) to A(z) by the identical algorithm.

We will now prove that the extension of the autocorrelation function given by (II-34) does indeed correspond to the maximum entropy spectrum by showing that

$$H(z) + H^*(z^{-1}) = \frac{R(0) \prod_{n=1}^{N} (1-|c_n|^2)}{A(z)\, A^*(z^{-1})}, \qquad \text{(II-42)}$$

which is a combination of (II-13) and (II-20). The proof again uses induction and is illustrated by going from N-1 to N .

If we let A(z) and D(z) be the N-1th order polynomials, then if (II-42) is true for the N-1th order case, we discover that

$$\frac{R(0) \prod_{n=1}^{N-1} (1-|c_n|^2)}{A(z)\, A^*(z^{-1})} = \frac{R(0)}{2}\left[\frac{D(z)}{A(z)} + \frac{D^*(z^{-1})}{A^*(z^{-1})}\right] = \frac{R(0)}{2}\,\frac{D(z)A^*(z^{-1})+D^*(z^{-1})A(z)}{A(z)\, A^*(z^{-1})}$$

or

$$D(z)\, A^*(z^{-1}) + D^*(z^{-1})\, A(z) = 2 \prod_{n=1}^{N-1} (1 - |c_n|^2) . \qquad \text{(II-43)}$$

The Nth order case can be written in terms of A(z) , D(z) and c_N as

$$\frac{R(0)}{2}\left[\frac{D(z) - c_N z^N D^*(z^{-1})}{A(z) + c_N z^N A^*(z^{-1})} + \frac{D^*(z^{-1}) - c_N^* z^{-N} D(z)}{A^*(z^{-1}) + c_N^* z^{-N} A(z)}\right] \qquad \text{(II-44)}$$

$$= \frac{R(0)}{2}\,\frac{[D(z)A^*(z^{-1}) + D^*(z^{-1})A(z)] - |c_N|^2[D(z)A^*(z^{-1}) + D^*(z^{-1})A(z)]}{[A(z) + c_N z^N A^*(z^{-1})][A^*(z^{-1}) + c_N^* z^{-N} A(z)]},$$

where the numerator terms in c_N and c_N^* vanish. Using (II-43), this becomes our desired result of

$$\frac{R(0) \prod_{n=1}^{N} (1 - |c_n|^2)}{[A(z) + c_N z^N A^*(z^{-1})][A^*(z^{-1}) + c_N^* z^{-N} A(z)]} . \qquad \text{(II-45)}$$

Since (II-42) is trivially true for the zeroth order case, our inductive proof is finished.

In this section, we have proved the sufficiency condition of the Fundamental Autocorrelation Matrix Theorem in the case of a positive definite matrix by actually constructing a power spectrum which agrees with the autocorrelation values. Furthermore, we have shown that the infinite extension of the autocorrelation function corresponding to setting all the rest of the reflection coefficients to zero is generated by using the prediction error filter in a feedback operation. The fact that the prediction error filter is minimum phase is necessary and sufficient for the extension to be stable. Finally, since we have constructed the maximum entropy spectrum from the autocorrelation lag values by a straightforward algebraic procedure, all of the general questions that were raised about the variational derivation can be answered in the positive.

8. Pure Spectral Line Extensions

Our previous section treated the case when the finite autocorrelation matrix was positive definite. Here we shall look at the case when the finite matrix is strictly non-negative definite.

Since we have already seen in section II-B-4 that the extension of the autocorrelation function is uniquely determined once the matrix becomes singular, we shall concentrate our attention at the transition point when the N by N matrix is positive definite but the N+1 by N+1 matrix is singular. When this happens, we have already seen that $|c_N|$ will be unity and the mean square error, P_N, will be equal to zero. Also, if $A(z)$ is the N-1 th order prediction error filter, then the N th order filter is $A(z) + c_N z^N A^*(z^{-1})$. Since $P_N=0$, this filter does a perfect job of predicting the next point in the time series. We shall now show in two ways that this perfect prediction error

filter is not minimum phase but instead has all of its zeros on the unit circle.

For the first proof, suppose that we replace c_N by αc_N where α is real and slightly less than unity. Then all N roots of $A(z) + \alpha c_N z^N A^*(z^{-1})$ are outside the unit circle. Next suppose α is real and slightly greater than unity. Then $|\alpha c_N z^N A^*(z^{-1})| > |A(z)|$ for $|z| = 1$ and since $z^N A^*(z^{-1})$ has all of its N roots inside the unit circle, so does $A(z) + \alpha c_N z^N A^*(z^{-1})$. Thus, as α goes from slightly less than unity to slightly more than unity, the N roots go from outside to inside the unit circle. Since the roots of a polynomial are continuous functions of the coefficients of the polynomial, we see that when $\alpha=1$, all of the roots must lie on the unit circle.

The second proof simply relies on the fact that as z goes around the unit circle once, the phase of $A(z)$ varies but does not have any net phase shift. However, at the same time, the phase of $c_N z^N A^*(z^{-1})$ makes a total of N complete revolutions. Since $|A(z)| = |c_N z^N A^*(z^{-1})|$, and the phases are continuous functions of z, there must be at least N places on the unit circle where $A(z)$ and $c_N z^N A^*(z^{-1})$ are opposite in phase. Since the prediction error filter is of order N, these N places are all of the zeros of the filter. This second proof also shows that the N roots are distinct, i.e., there are no double or higher order zeros.

Since the mean square error is zero, the output spectrum must be zero at all frequencies. This means that the input spectrum can be non-zero only where the power response of the filter is zero, namely at the N zeros of the filter. Furthermore, there must be power at each of these

N frequencies, since otherwise a shorter prediction error filter would have been able to do perfect prediction. However, the amount of power at each of the N frequencies cannot be found from the Nth order prediction error filter since this filter is the same perfect predictor independent of how the power is distributed at the N frequencies. This is curious since the sequence of prediction error filters up to the Nth depends on the amount of power in the N delta functions, but the Nth filter is the same for all such sequences. This, however, agrees with our observation in section II-A-5, (II-21), that when $|c_N|=1$, we cannot generate the N-1th order filter from the Nth order filter.

One way of finding the power in the delta functions is to realize that the autocorrelation function must be expressible as

$$R(n) = \sum_{m=1}^{N} r_m e^{-i2\pi f_m n\Delta t} ,$$

where the r_m are positive and the f_m are the delta function frequencies. Knowing R(n) , n=0 to N-1 , allows one to solve for the r_m . One method is to form the N-1th order filter

$$\frac{\prod_{m=1}^{N} (1-\alpha_m z)}{1-\alpha_n z} ,$$

where $\alpha_m = e^{i2\pi f_m \Delta t}$ and do a dot product into the autocorrelation values. Since this filter has zero response to each delta function except the nth one, the strength of the nth delta function can be determined from this dot product. A much deeper study of the properties of the delta functions is given in Appendix A.