Velocity Estimation Recapitulated

by Jon Claerbout

This section is a chapter from my forthcoming book, _Fundamentals of Geophysical Data Processing_ (McGraw-Hill, 1976). My teaching experience has enabled me to make a considerable improvement in presentation over SEP vol. 2, p. 316-320. Of course the definitive work in this area is Stephen M. Doherty's PhD thesis (SEP vol. 4).

11 - 3. Velocity Estimation

Previous chapters focused on the task of delineating earth structure. Mathematically this has meant that we have taken the material velocity known (and for convenience constant) but the impedance to have unknown discontinuities at interfaces of unknown shape between geologic structures. Now we seek to find the material velocity. Traditionally this has been done by assuming the earth structure consists of plane horizontal layers. Then the material velocity is deduced from the offset dependent time shift (called the normal moveout correction or NMO) which best flattens the events on the common midpoint gathers. In the present chapter it will be shown how the assumption of flat layers may be eliminated. We will see how velocity can be estimated even in an earth consisting of random point scatterers. This can be expected to be useful in fractured zones or even perhaps in "no record" areas. An "NR" or "no record" area is where the best processed section shows no coherence along the midpoint y coordinate. An area may be NR not only because of poor data quality but also because the geologic structure itself has no continuity. But, as we will see, there is no theoretical reason why material velocity cannot be determined in such an NR area.

Basically the procedure is to downward continue both the theoretical downgoing wave and the observed upgoing wave. They are projected back down to the reflectors where their nearly constant ratio should represent the reflection coefficient as a function of offset. If they are projected downwards with an incorrect velocity, the ratio will be an oscillatory function of offset. The task then is to find the velocity which gives the best fit of the two waves. It doesn't matter whether the reflectors have any lateral continuity or not because the fitting is done for variable offset at a fixed midpoint at the reflector depth. When reflectors have no lateral continuity they may be called scatterers. An earth model with randomly located scatterers would produce migrated seismic data which was a random function of (moveout corrected) time and midpoint but which was a constant function of offset.

It is easy to think of a good means to downward continue the the downgoing waves. From the shot point these waves expand spherically. For a homogeneous medium we can just write down an analytic solution. For a moderately inhomogeneous medium we can use the methods of earlier chapters. One problem is that the approximation $Q_{zz} \approx 0$ restricts validity to angles of about 15° from the vertical. This is easily improved by transforming from cartesian (x,z) coordinates to polar (r,θ) coordinates. The approximation $Q_{rr} \approx 0$ requires rays to stay within 15° of a radius line. Obviously a "stratified media coordinate frame" could be designed to handle even stronger velocity inhomogeneity of that type.

The problem which is more difficult is to find a good coordinate system for the upcoming waves. It took me two years to come up with a practical solution. A hint is provided by observing why, for the downgoing wave, the polar system is preferable to the cartesian system. For a quasi-spherical wave $Q_{\theta\theta}$ will be nearly zero, whereas Q_{xx} gets big quickly unless you are directly under the source. Because we deal with equations like $Q_{zt} = Q_{xx}$ or $Q_{rt} = Q_{\theta\theta}/r^2$ this means that Q_r will generally be small, whereas Q_z is small only on the z-axis directly under the source. Consequently the approximation $Q_{rr} \approx 0$ is much better than $Q_{zz} \approx 0$. Our observation is that the advantage of the (r,θ) coordinates is that the downgoing wave $D(r,\theta)$ is nearly independent of the lateral θ coordinate. What we need is a coordinate frame in which the upcoming wave U is nearly independent of the lateral coordinate. Experienced geophysicists will immediately recognize that normal moveout corrected data fills this requirement. Normal moveout (NMO) correction is a compression of the time axis on far offset seismograms intended to make the far offset waves arrive at the same (NMO corrected) time as the vertically incident waves. Thus, the partial derivative of the wave field Q with respect to shot-geophone offset at a fixed NMO corrected time should be small.

The closer our data comes to that from flat horizontal reflectors in an earth of known velocity the smaller the offset derivative will be. The purpose of a wave equation is to handle the departure from such an idealized situation.

This compression of the time axes of the far offset seismograms is really a coordinate change. The usual definition of NMO correction does not anticipate our desire to project our geophones deeply into the earth. As we project our geophones downward along a ray path we will retain the <u>surface midpoint</u> y and the <u>surface half offset</u> $h = f/2$ as lateral coordinates of the wave field. Lateral derivatives of idealized data should vanish.

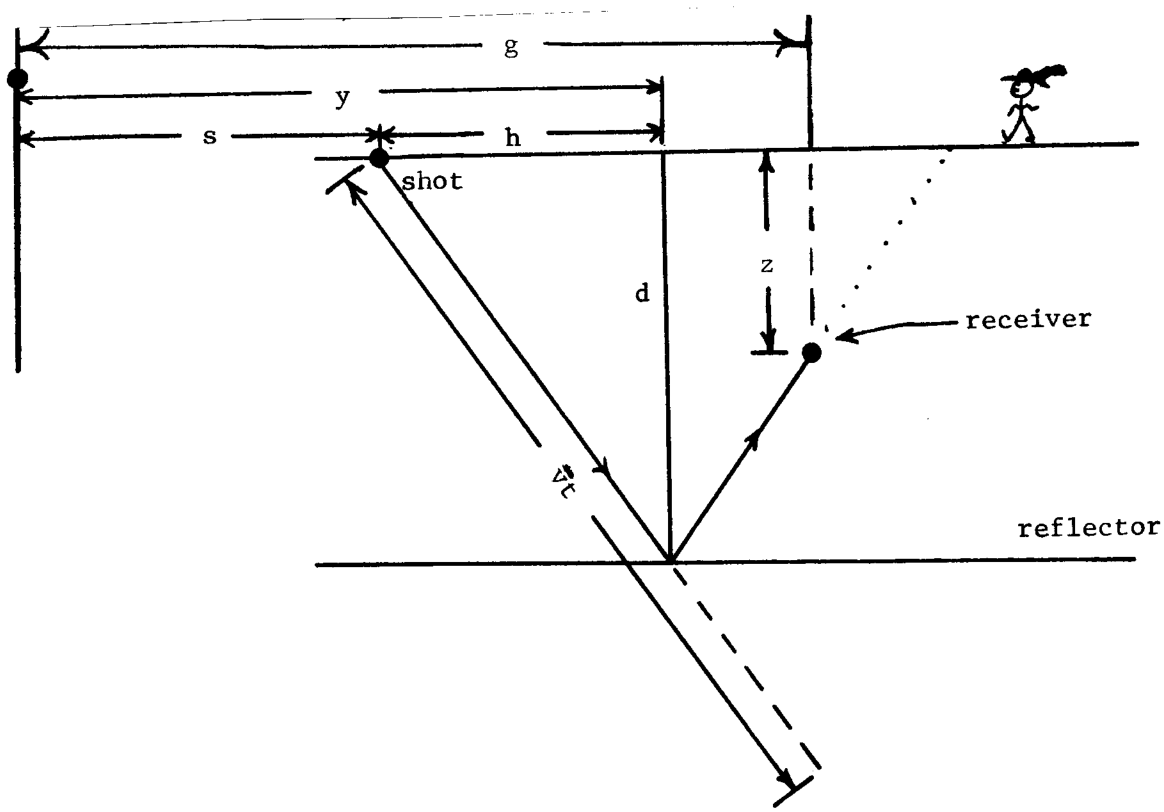


Figure 11-14. Geometry for normal moveout correction of downward continued data.

Figure 14 shows the geometry for normal moveout correction of downward continued data in homogeneous media of velocity $\bar{v}$. The transformation from interpretation variables to observation variables is

$$s(h, y, d, z) = y - h \quad (1a)$$

$$g(h, y, d, z) = y + (d-z)\,h/d \quad (1b)$$

$$t(h, y, d, z) = (d^2 + h^2)^{1/2}(2d - z)/(d\,\bar{v}) \quad (1c)$$

Either algebraic or geometric means yield the inverse transformation

$$d(s, g, t, z) = \frac{1}{2}[(\bar{v}^2 t^2 - (g-s)^2)^{1/2} + z] \tag{2a}$$

$$y(s, g, t, z) = \frac{1}{2}[(g+s) + \frac{z(g-s)}{(\bar{v}^2 t^2 - (g-s)^2)^{1/2}}] \tag{2b}$$

$$h(s, g, t, z) = \frac{1}{2}[(g-s) + \frac{z(g-s)}{(\bar{v}^2 t^2 - (g-s)^2)^{1/2}}] \tag{2c}$$

That (2) is indeed inverse to (1) is readily checked by substituting (1) into (2).

In a homogeneous medium of velocity $\tilde{v}$ we may write the solution for the downgoing wave as a delta function on an expanding circle

$$D(g, s, t, z) = \delta((g-s)^2 + z^2 - \tilde{v}^2 t^2) \tag{3}$$

The upcoming wave U will be computed in the (h, y, d, z) variables and we want to compare it to the downgoing wave D, expressed by (3) in (g, s, t, z) variables. We can convert D to (h, y, d, z) variables by substitution of (1) into (3); a meaningful simplification arises if we assume the medium velocity $\tilde{v}$ equals the moveout coordinate frame velocity $\bar{v}$. We get

$$D(h, y, d, z) = \delta(4d(z-d)) \tag{4}$$

In this case the downgoing wave turns out to be independent of the lateral coordinates h and y.

Now let us consider an earth model which contains only a single point scatterer located at (x_0, z_0). This scatterer is illuminated by a delta function source located at $(s,0)$. Excluding horizontally propagating waves we have for the upcoming wave $U(s,g,t,z)$ an infinitesimal distance above the scatterer

$$U(s,\ g,\ t,\ z_0 - 0) = \delta(g-x_0)\ \delta(\bar{v}^2 t^2 - (s-x_0)^2 - z_0^2) \tag{5}$$

Substituting the transformation (1) at $z=d$ into (5) we obtain

$$U(h,\ y,\ d,\ z=d) = \delta(y-x_0)\ \delta(d^2 + h^2 - (y-h-x_0)^2 - z_0^2)$$

The existance of $\delta(y-x_0)$ allows us to set $y=x_0$ in the other delta function getting

$$U(h,\ y,\ d,\ z=d) = \delta(y-x_0)\ \delta(d^2 - z_0^2)$$

We now see the central concept that the wave at the reflector in moveout corrected coordinates is indeed independent of the half-offset h. Obviously the superposition of a random collection of point scatterers will create a migrated wave field which is random in y and d but still constant in the offset h. Indeed the concept would also seem to be valid even if the scatterers were randomly distributed out of the plane of the section. In three dimensional space it is only necessary to regard z as the radial distance from the traverse line.

The purpose of all this is to estimate velocity, but velocity is needed for the first step, namely the migration. Use of an erroneous velocity in the migration prevents total collapse to a delta function on the midpoint axis. This causes some destructive interference between adjoining midpoints representing some information loss for a random scatterer model but it is of no consequence in a layered earth model (where even the migration itself is unnecessary).

Stephen M. Doherty [Ref. 37] made a calculation to illustrate these concepts. Figure 15 shows an earth model. Figure 16 shows surface data and downward continued data for the model.

From the point of view of velocity determination, it is immaterial what coordinate frame is used to downward continue the observed waveforms. However, it is convenient to downward continue these waves in the NMO coordinate frame. This proceeds in a fashion similar to our earlier work. To simplify the algebra, first note that (2b) and (2c) imply that

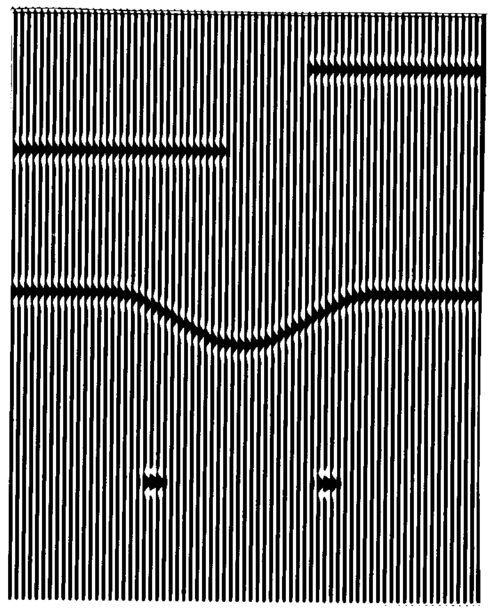
$$\frac{\partial y}{\partial(g,t,z)} = \frac{\partial h}{\partial(g,t,z)} \qquad (6)$$

The wave equation

$$(\partial_{gg} + \partial_{zz} - \partial_{tt} / \tilde{v}^2) P = 0 \qquad (7)$$

in NMO coordinates will take the form

z=2.5



z=3.5

Figure 11-15. An earth model used to illustrate velocity analysis with downward continued data.

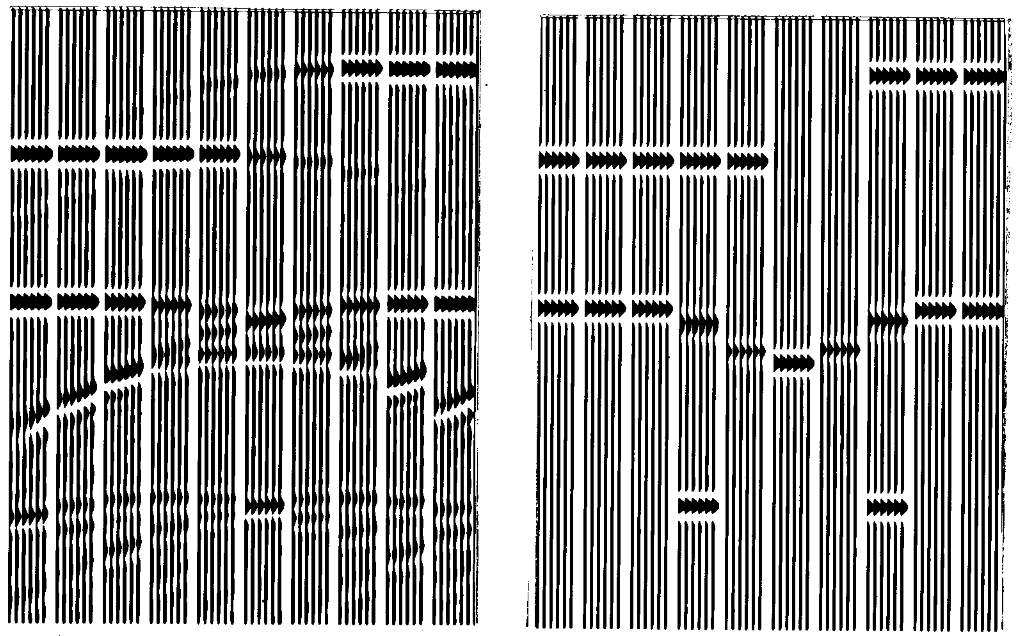


Figure 11-16. Surface data and downward continued data for the model of Figure 15. The coordinates are designed to display three dimensional data (y,h,d) on a two dimensional page. The vertical axis, as usual, is the d coordinate. For the horizontal axis the h coordinate has been sampled at 6 values of h which are displayed together in groups (common midpoint gathers). There are 10 of these gathers spaced along the y-axis. Within each group h=0 is on the left and h_{max}, corresponding to about 45° rays, is on the right. The left frame shows the surface data and the right frame shows the data down at the reflectors. At the reflectors we see horizontal alignment of waveforms indicating that the data is independent of offset h .

$$
\begin{aligned}
[\quad & (d_g \partial_d + y_g \partial_y + h_g \partial_h)^2 + \\
& + (\partial_z + d_z \partial_d + y_z \partial_y + h_z \partial_h)^2 - \\
& - (d_t \partial_d + y_t \partial_y + h_t \partial_h)^2 / \tilde{v}^2 \,] \, Q = 0
\end{aligned}
\tag{8}
$$

As before, when we square these partial differential operators we will take the coefficients to be constant. This is the high frequency assumption that the wave field changes more rapidly than the coordinate frame. Before we compute all the required derivatives we define a simplifying combination b where

$$
b = (\bar{v}^2 t^2 - (g - s)^2)^{1/2} \tag{9}
$$

The required derivatives are computed recalling (6) to be

$$
\begin{bmatrix} d_g & d_z & d_t \\ y_g & y_z & y_t \\ h_g & h_z & h_t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -(g-s)/b & 1 & \bar{v}^2 t/b \\ 1+z\bar{v}^2 t^2/b^3 & (g-s)/b & -(g-s)\bar{v}^2 tz/b^3 \\ 1+z\bar{v}^2 t^2/b^3 & (g-s)/b & -(g-s)\bar{v}^2 tz/b^3 \end{bmatrix} \tag{10}
$$

First we quickly discover that if moveout correction velocity $\bar{v}$ equals media velocity $\tilde{v}$, say $\bar{v} = v$, then three of the terms in (8) vanish identically. By direct substitution the reader may verify that

$$
(d_g^2 + d_z^2 - d_t^2 / v^2) \, Q_{dd} = 0 \tag{11a}
$$

$$
2 (d_g y_g + d_z y_z - d_t y_t / v^2) \, Q_{dy} = 0 \tag{11b}
$$

$$
2 (d_g h_g + d_z h_z - d_t h_t / v^2) \, Q_{dh} = 0 \tag{11c}
$$

Next we obtain the three cross terms with ∂_z .

$$2\ y_z\ Q_{yz} = (g-s)/b\ Q_{yz} = h/d\ Q_{yz} \tag{12a}$$

$$2\ h_z\ Q_{hz} = (g-s)/b\ Q_{hz} = h/d\ Q_{hz} \tag{12b}$$

$$2\ d_z\ Q_{dz} = Q_{dz} \tag{12c}$$

From (6) we realize that the coefficients of Q_{yy} , Q_{hh} , and $2\,Q_{yh}$ are identical. Through a considerable amount of algebraic reduction we obtain

$$(y_g^2 + y_z^2 - y_t^2 / v^2)(\partial_y + \partial_h)^2 Q = -\left(\frac{d}{2d-z}\right)^2 (1 + h^2/d^2)(\partial_y + \partial_h)^2 Q \tag{13}$$

As usual we make the Fresnel-like approximation by dropping the Q_{zz} term. In cartesian geometry this limits accurate treatment of rays to within a cone of about 15 degrees of the vertical. In the NMO geometry this would seem to be more like a 15° limitation on structural dips. Of course the higher accuracy techniques can always be used where required. Gathering (11) to (13) together we obtain the basic result

$$(\partial_d + h/d(\partial_y+\partial_h))\,\partial_z Q = -\left(\frac{d}{2d-z}\right)^2 (1 + h^2/d^2)(\partial_y + \partial_h)^2 Q \tag{14}$$

Equation (14) may be used for downward continuation of moveout corrected unstacked sections for velocity determination.

It seems worthwhile to inspect (14) in some special cases. At the surface for zero offset Q_h vanishes by symmetry. For idealized data from layered reflectors Q is a function of d only. In a wide variety of practical situations it turns out to be reasonable to simplify (14) with $Q_d >> Q_y >> Q_h$. This leaves us with

$$Q_{dz} = -\left(\frac{d}{2d-z} \right)^2 \left(1 + \frac{h^2}{d^2}\right) Q_{yy} \qquad (15)$$

It seems natural to wonder about the variable coefficient $d / (2d-z)$ in comparison to the earlier equations with constant coefficients. We can now show that with regard to migration that there is no practical difference. Define a new variable

$$z' = z\,d / (2d-z) \qquad (16)$$

Note that at the surface $z=0$ we have z' equal zero and at the reflectors $z=d$ we have $z' = d$. Thinking of $Q(z,d) = Q'(z',d)$ we find

$$Q_z = z'_z \, \partial_{z'} \, Q'$$

$$Q_d = (\partial_d + z'_d \, \partial_{z'}) \, Q'$$

With these the left side of (15) becomes

$$Q_{dz} = (\partial_d - z'_d \, \partial_{z'}) \, z'_z \, \partial_{z'} \, Q$$

In a Fresnel-like approximation we drop $\partial_{z'z'}$ obtaining

$$Q_{dz} = z'_z \, Q_{dz'} = \frac{2d^2}{(2d-z)^2} \, Q_{dz'}$$

which reduces (15) to

$$Q_{dz'} = -\frac{1}{2} \left(1 + \frac{h^2}{d^2}\right) Q_{yy} \qquad (17)$$

To justify the factor of two which was asserted in chapter 11-2 we may make another transformation from d to a two way travel time coordinate t' given by

$$t' = 2d/v$$

which gives

$$Q_{t'z'} = -\frac{v}{4}\left(1 + \left(\frac{2h}{vt'}\right)^2\right) Q_{yy} \qquad (18)$$

Of course (18) must be integrated from $z'=0$ to $z'=t'v'/2$. A convenient rescaling of the depth axis is in terms of two way travel time t'' where

$$t'' = 2z'/v$$

This leads to the equation

$$Q_{t't''} = -\frac{v^2}{8}\left(1 + \left(\frac{2h}{vt'}\right)^2\right) Q_{yy} \qquad (19)$$

in which t' is the two way travel time and t'' is the two way travel time depth axis which is integrated from the surface $t'' = 0$ to the reflectors at $t'' = t'$.

Strictly speaking (19) should be applied separately to data of each offset h before data is summed over offset (stacked). For reasons of economy the data is often stacked before migration with (19). In such a compromise h in (19) is taken zero or some average value of $2h/vt'$ is used.

So far we have shown that downward-continued, moveout-corrected seismograms will be independent of offset if downward continued with the correct velocity. What we have not seen is how to estimate the velocity error from the downward continued data. For this we must recognize another important term which has been omitted from the entire analysis. We saw this term in earlier studies of propagation in inhomogeneous media. We must carry through the distinction between media velocity $\tilde{v}(x,z)$ and NMO velocity $\bar{v}$ (generalizable to $\bar{v}(z)$) which was abandoned for the sake of the simplifications beginning at (11). Recalling that for small departures from layered models, $Q_d >> Q_y >> Q_h$. We see that the first of the three terms in (11) will be the most important. Making the distinction between the two velocities equation (11a) now introduces the significant term

$$(d_g^2 + d_z^2 - d_t^2 / \tilde{v}^2) Q_{dd} \neq 0$$

$$\frac{\bar{v}^2 t^2}{\bar{v}^2 t^2 - (g-s)^2} \left(1 - \frac{\bar{v}^2}{\tilde{v}^2} \right) Q_{dd} \neq 0$$

$$\left(1 + \frac{h^2}{d^2} \right) \left(1 - \frac{\bar{v}^2}{\tilde{v}^2} \right) Q_{dd} \neq 0 \quad (21)$$

Thus, with this new term but the other approximations equation (15) becomes

$$Q_{dz} = - \left(\frac{d}{2d - z} \right)^2 \left(1 + \frac{h^2}{d^2} \right) Q_{yy} + \left(1 + \frac{h^2}{d^2} \right) \left(1 + \frac{\bar{v}^2}{\tilde{v}^2} \right) Q_{dd} \quad (22)$$

Numerically we can consider solving (22) by a splitting method where the solution is projected downward by alternate use of the two equations

$$Q_{dz} = - \left(\frac{d}{2d - z} \right)^2 \left(1 + \frac{h^2}{d^2} \right) Q_{yy} \quad (23a)$$

$$Q_z = \left(1 + \frac{h^2}{d^2} \right) \left(1 - \frac{\bar{v}^2}{\tilde{v}^2} \right) Q_d \quad (23b)$$

Equation (23a) may be called the "diffraction" part and (23b) may be called the "thin-lens" part. The effect of (23b) is that as Q is projected in the z-direction, each seismogram (a seismogram is a function of (moveout corrected) time d at a fixed half offset h

and midpoint y) undergoes a steady time shift (d shift). The amount of the shift increases with the velocity error according to $(1 - \bar{v}^2/\tilde{v}^2)$ and it increases with offset according to $(1 + h^2/d^2)$. Thus, the effect of (23b) is to change the curvature of the data with half offset h. However (23a) contains Q_{yy} but it does not contain Q_h or Q_{hh}. This means that the operations of (23a) and (23b) commute. Thus, we can project all the way down to the reflectors with (23a) and then use (23b). It is significant that the hard part of the job, namely (23a), depends on the frame velocity $\bar{v}$, not the material velocity $\tilde{v}$. This means that we can rather economically test various media velocities $\tilde{v}$.

Before we can consider the task of selecting our best estimate of the media velocity $\tilde{v}$, we must consider the matching of the upcoming wave U to some reflection coefficient c times some downgoing wave D. The matching of these waves can be done in the field recording coordinates but we prefer to do the matching in the NMO coordinate system. First let us get an expression for the downgoing wave in NMO coordinates. Insert (1) into (3) to obtain

$$D(h, y, d, z) = \delta \left[4d(z-d) + \left(1 - \frac{\tilde{v}^2}{\bar{v}^2}\right)\left(1 + \frac{h^2}{d^2}\right)(2d - z)^2 \right] \qquad (24)$$

At present we are not trying to preserve slow magnitude variations (spherical spreading was omitted from (3)), so we can divide through the argument of the delta function by $-4d$. Since we are interested in small amounts of variation of $\tilde{v}$ from $\bar{v}$ the delta function will vanish very near to $z = d$. Thus, to a good approximation we can substitute z for d in the coefficient of $(1 - \tilde{v}^2 / \bar{v}^2)$ obtaining

$$D(h, y, d, z) = \delta [d - z - (1 - \frac{\tilde{v}^2}{\bar{v}^2}) (1 + \frac{h^2}{z^2}) \frac{z}{4}] = \delta (d - z - s) \tag{25}$$

where we have defined a time (d) shift function

$$s(h^2, z, \tilde{v}/\bar{v}) = (1 - \frac{\tilde{v}^2}{\bar{v}^2})(1 + \frac{h^2}{z^2}) \frac{z}{4} \tag{26}$$

Now let us return to the task of matching the up and downgoing wave. We might hope to determine a reflection coefficient, along with some angular dependence, in the form of a power series, for example

$$c = c_0 + c_1 h/z + c_2 h^2/z^2 + \dots \tag{27}$$

To simplify the sequel we will estimate only the constant term c_0 by the minimization

$$\min_c \sum_h \sum_d [U(y, z, h, d) - c(y,z) D(y, z, h, d)]^2 \tag{28}$$

The solution is obviously

$$c(y,z) = \frac{\sum_h \sum_d U D}{\sum_h \sum_d D^2} \tag{29}$$

Because D vanishes almost everywhere we can gain insight by replacing the double sum by a single sum, specifically for the numerator

$$\text{Numerator} = \sum_h \sum_d U(y, z, h, d) \delta(d - z - s) = \sum_h U [y, z, h, d = z + s(h^2, z, \tilde{v}/\bar{v})] \tag{30}$$

Letting N denote the number of terms in the offset sum, we get for (29)

$$c(y, z, \tilde{v}) = \frac{1}{N} \sum_h U \tag{31}$$

Finally, we come to the part of determining the velocity $\tilde{v}$ which provides the best minimum of $U - c\,D$. For this a computer scan over $\tilde{v}$ may be used to find the minimum

$$\min_{\tilde{v}} \sum_{h} \sum_{d} (U - c\,D)^2$$

$$= \min_{\tilde{v}} \sum_{h} (U - c)^2$$

$$= \min_{\tilde{v}} \sum_{h}^{N} (U - \frac{1}{N} \sum_{h}^{N} U)^2$$

$$= \min_{\tilde{v}} (\sum_{h} U^2) - \frac{1}{N} (\Sigma U)^2 \geq 0 \qquad (32)$$

In practice it is found that rather than minimize the sum squared minus the squared sum it is preferable to maximize the negative logarithm or the semblance ratio

$$\underset{\tilde{v}}{\text{Semblance}} = \frac{(\Sigma U)^2}{N \Sigma U^2} \leq 1 \qquad (33)$$

The ratio has the advantage of being insensitive to the magnitude of the wave U and lends itself well to displays over a wide range of conditions.