SEPARATION OF THE FORWARD AND BACKWARD SOLUTIONS OF THE ONE-DIMENSIONAL WAVE EQUATION

R.S. Anderssen, Computer Centre
Australian National University
Canberra, Australia

ABSTRACT

A number of authors (see, for example, Corones [1] and Sluijter [2]) have discussed the significance of the decomposition of the solution of the wave equation into forward and backwards (up and down) wave solutions. In particular, in the context of numerical holography, Claerbout and co-workers have shown that the actual numerical separation is an essential requirement, and have used wave equation migration for this purpose (see, for example, Claerbout [3]). The possibility of using variational methods as a basis for the separation does not appear to have been investigated. In this report, we show that certain variational formulations for the wave equation can be used to separate computationally these two basic types of solutions. The actual separation depends heavily on the choice of coordinate functions for the variational solution.

1. INTRODUCTION AND PRELIMINARIES

In this report, we consider the problem of separating numerically, through the use of variational methods, the forward and backward solutions of the one-dimensional wave equation.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad u = u(x,t), \quad 0 \le x \le \ell, \quad 0 \le t < \infty, \tag{1}$$

along with the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x), \quad 0 \le x \le \ell, \tag{2}$$

and the boundary conditions

$$u(0,t) = u(\ell,t) = 0, \quad 0 \le t < \infty, \tag{3}$$

where, for notational simplicity, we have taken the velocity $c = 1$.

It is well known that independent variational formulations can exist for the solution of the same problem. In the case of the wave equation formulation given above, three such variational formulations are:

1. Hamilton's Principle. Let K_T denote the set of functions which satisfy (3) as well as

$$u(x,0) = f(x), \quad u(x,T) = u_T(x), \quad x \varepsilon R,$$

where $u_T(x)$ is prescribed. Hamilton's Principle asserts that, if $u \varepsilon K_T$, then

$$\delta\{ \frac{1}{2} \int_0^T \int_0^\ell [(\frac{\partial u}{\partial x})^2 - (\frac{\partial u}{\partial t})^2](x,t)\, dx\, dt\} = 0, \tag{4}$$

if and only if u satisfies (1), (2) and (3) on $[0,\ell] \times [0,\infty]$, where $\delta\{F\}$ denotes the first variation of the functional F.

2. <u>The Petrov-Galerkin Principle</u> (See Yaskova and Yakovlev [4]). Let K denote the set of functions which satisfy (2) and (3). Then, for a suitable set of functions $\eta(x,t)$ in K,

$$\int_0^T \int_0^\ell \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} \right) \eta(x,t)\, dx\, dt = 0 , \tag{5}$$

if and only if u satisfied (1), (2) and (3).

3. <u>The Gurtin Formulations</u>. (See Gurtin [5]). Let K denote the set of functions which satisfy the boundary conditions (3). For each $t \varepsilon (0,\infty)$ define the functional

$$\Delta_t(u) = \int_R [u*u + G* \frac{\partial u}{\partial x} * \frac{\partial u}{\partial x} + 2\, F * u]\, (x,t)dx \tag{6}$$

where

$$\begin{aligned} G(x,t) &= t, \\ F(x,t) &= -f(x) - tg(x), \\ u*v &= \int_0^t u(x,t-\tau)\, v(x,\tau)d\tau. \end{aligned} \tag{7}$$

Then, for $u \varepsilon k$,

$$\delta\Delta_t(u) = 0 \qquad (0 \leq t < \infty) ,$$

if and only if u is a solution of the wave equation (1), (2) and (3).

It is clear that Hamilton's Principle is not applicable, since it presupposes a knowledge of the function u at a later time T - something which is unknown in advance. The Petrov-Galerkin principle circumvents this difficulty through the additional flexibility contained in the choice of the $\eta(x,t)$ functions and it will be this variational formulation which we shall examine in this report.

In Section 2,we show how to construct a variational solution of (1), (2) and (3) via the Petrov-Galerkin formulation. Then, in Section 3, a procedure based on this is used to separate out the forward and backward components.

For clarity, we examine the formulation (1), (2) and (3) which describes the motion of a plucked string in which $g(x) = 0$. In this case, the exact solution is sknown (see, Lanczos [6], Section 8.7) as a Fourier series solution:

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} C_R \cos \frac{k\pi}{\ell} t \quad \sin \frac{k\pi}{\ell} x \\ &= \frac{1}{2} \sum_{k=1}^{\infty} C_R \left[\sin \frac{k\pi}{\ell} (x+t) + \sin \frac{k\pi}{\ell} (x-t)\right] \qquad (8) \\ &= \frac{1}{2} \left[f(x+t) + f(x-t)\right] , \end{aligned}$$

where

$$C_k = \frac{2}{\ell} \int_0^{\ell} f(\varepsilon) \sin \frac{k\pi}{\ell} \varepsilon \, d\varepsilon$$

The forward and backward wave motions which make up the solution are clearly discernible in the above representations of the exact solutions.

2. Direct Variational Solution

The approximate minimization of a functional F(u) or the approximate solution of the Petrov-Galerkin equations can be performed as a two stage procedure:

STAGE 1. Choose a system of coordinate (trial) functions $\{\phi_k(x,t)\}$ which satisfy the following conditions:

(i) The system $\{\phi_k(x,t)\}$ is complete in some appropriate Hilbert space such as the space of square summable functions defined on $[0,\ell] \times [0,\infty]$.

(ii) Each member of the system $\{\phi_k(x,t)\}$ satisfies the homogeneous boundary conditions (3).

(iii) For arbitrary finite n, the set $\phi_1(x,t)$, $\phi_2(x,t)$, ---, $\phi_n(x,t)$ is linearly independent.

STAGE 2. Using the following approximations for the soultion u, viz.

$$u_n = \sum_{k=1}^{n} a_k^{(n)} \phi_k(x,t) , \qquad k = 1, 2, ------, \tag{10}$$

determine the unknowns $a_k^{(n)}$ so as to either minimize $F(u_n)$, or solve the Petrov-Galerkin equations with respect to an appropriate choice for the $\eta(x,t)$; for example, $\eta(x,t) = \phi_k(x,t)$, k = 1,2, --- .

The first condition of STAGE 1 ensures that approximations of the form (10) can approximate the desired solution arbitrarily closely. The second ensures that all approximations satisfy the boundary conditions, while the third is required to ensure that the resulting system of linear equations which determine the $a_k^{(n)}$ (k=1,2,---,n) are non-singular

Thus, the nature of the approximations will depend heavily on the choice of the $\{\phi_i\}$, and the $\eta(x,t)$ in the case of the Petrov-Galerkin equations. For example, Mikhlin [7] has shown that the choice of a strongly minimal system for the $\{\phi_k(x,t)\}$ is a necessary and sufficient condition for the stable computational implementation of the above direct procedure. Below, we shall show that the separation of the solution of the wave equation (1), (2) and (3) with $g(x) = 0$ into forward and backward components can be achieved through an appropriate choice of the $\phi_i(x,t)$ and the $\eta(x,t)$.

Here we show how the Petrov-Galerkin equations can be used to obtain the Fourier series solution of (1), (2) and (3) with $g(x) = 0$. Initially, we integrate (5) by parts twice to obtain

$$(u, \frac{\partial^2 \eta}{\partial x^2}) - \int_0^T [\frac{\partial u}{\partial x} \eta - u \frac{\partial \eta}{\partial x}]_{x=0}^{x=\ell} dt$$

$$= (u, \frac{\partial^2 \eta}{\partial t^2}) - \int_0^T [\frac{\partial u}{\partial t} \eta - u \frac{\partial \eta}{\partial t}]_{t=0}^{t=T} dx .$$

Assuming that the $\eta(x,t)$ satisfy the same boundary conditions as the $\phi_i(x,t)$, and using the fact that $g(x) = 0$, the Petrov-Galerkin equations become

$$(u, \frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial t^2}) = \int_0^\ell [u \frac{\partial \eta}{\partial t}]_{t=0}^{t=T} dx - \int_0^\ell [\frac{\partial u}{\partial t} \eta]_{t=T} dx \qquad (11)$$

Choosing the coordinate functions $\{\phi_k(x,t)\}$ to have the form

$$\phi_k(x,t) = \cos \frac{k\pi}{\ell} t \sin \frac{k\pi}{\ell} x , \quad k = 1, 2, 3, \ldots, \qquad (12)$$

and the $\eta(x,t)$ as the sequence

$$\eta(x,t) = \sin \frac{j\pi}{\ell} t \sin \frac{j\pi}{\ell} x, \quad j = 1, 2, 3, ---, \tag{13}$$

we obtain that:

(a) $$(u, \frac{\partial^2 \eta}{\partial x^2}) = -\sum_{k=1}^{n} a_k^{(n)} \int_0^{\ell} \int_0^{T} (j\pi/\ell)^2 \cos \frac{k\pi}{\ell} t \sin \frac{k\pi}{\ell} x \sin \frac{j\pi}{\ell} t$$

$$\sin \frac{j\pi}{\ell} xdxdt = (u, \frac{\partial^2 \eta}{\partial t^2}).$$

(b) $$\int_0^{\ell} [u \frac{\partial \eta}{\partial t}]_{t=0}^{t=T} - \int_0^{\ell} [\frac{\partial u}{\partial t} \eta]_{t=T} = -\int_0^{\ell} f(x) \; (\frac{j\pi}{\ell}) \sin \frac{j\pi}{\ell} xdx$$

$$+ \sum_{k=1}^{n} a_k^{(n)} \int_0^{\ell} (\frac{j\pi}{\ell}) \cos \frac{k\pi}{\ell} T \sin \frac{k\pi}{\ell} x \cos \frac{j\pi}{\ell} t \sin \frac{j\pi}{\ell} xdx$$

$$+ \sum_{k=1}^{n} a_k^{(n)} \int_0^{\ell} (\frac{j\pi}{\ell}) \sin \frac{k\pi}{\ell} T \sin \frac{k\pi}{\ell} x \sin \frac{j\pi}{\ell} t \sin \frac{j\pi}{\ell} x \, dx =$$

$$- \int_0^{\ell} f(x) \; (\frac{j\pi}{\ell}) \sin \frac{j\pi}{\ell} x \, dx + (\sin^2 \frac{j\pi}{\ell} T + \cos^2 \frac{j\pi}{\ell} T) \; a_j^{(n)} \; (\frac{j\pi}{\ell}) \frac{\ell}{2}$$

$$= \frac{-\ell}{2} (\frac{j\pi}{\ell}) \{ \frac{2}{\ell} \int_0^{\ell} f(x) \sin \frac{j\pi}{\ell} x \, dx - a_j^{(n)} \} .$$

Combining (a) and (b), we obtain that the approximate solution of the Petrov-Galerkin equations is

$$u_n = \sum_{k=1}^{n} a_k^{(n)} \cos \frac{k\pi}{\ell} t \sin \frac{k\pi}{\ell} x , \tag{14}$$

where

$$a_k^{(n)} = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\frac{j\pi}{\ell} x dx = C_k \tag{15}$$

which tends in the limit to the Fourier series solution (8).

3. Separation of the Forward and Backward Components

We now show there exists a choice for coordinate functions $\{\phi_k(x,t)\}$ and the $\eta(x,t)$ which allows the separation of the solution of the wave equation into forward and backward components. In fact, let

$$\phi_k(x,t) = \sin\frac{k\pi}{\ell}(x \pm t) \qquad (k = 1, 2, 3, ---) , \tag{16}$$

and

$$\eta(x,t) = \cos\frac{j\pi}{\ell}(x \pm t) \qquad (j = 1, 2, 3, ---) . \tag{17}$$

Then, we obtain that:

(a) $$(u, \frac{\partial^2\eta}{\partial x^2}) = -\sum_{k=1}^{n} a_k^{(n)} \int_0^{\ell}\int_0^{T} \sin\frac{k\pi}{\ell}(x \pm t)(j\pi/\ell)^2 \cos\frac{j\pi}{\ell}(x \pm t) dxdt$$

$$= (u, \frac{\partial^2\eta}{\partial t^2}) \qquad (j = 1, 2, ---, n) .$$

(b) $$\int_0^{\ell} [u \frac{\partial\eta}{\partial t}]_{t=0}^{t=T} dx - \int_0^{\ell} [\frac{\partial u}{\partial t}\eta]_{t=T} dx = \pm \{ \int_0^{\ell} f(x)(j\pi/\ell)$$

$$\sin\frac{j\pi}{\ell} xdx - \sum_{k=1}^{n} a_k^{(n)} \int_0^{\ell} \{ \sin\frac{k\pi}{\ell}(x \pm T)(j\pi/\ell)\sin\frac{j\pi}{\ell}(x \pm T)$$

$$+ (k\pi/\ell) \cos \frac{k\pi}{\ell} (x \pm t) \cos \frac{k\pi}{\ell} (x \pm t) \cos \frac{j\pi}{\ell} (x \pm T) \} \, dx$$

$$= \pm \frac{j\pi}{\ell} \{ \int_0^\ell f(x) \sin \frac{j\pi}{\ell} x \, dx - \ell a_j^{(n)} \}$$

Combining (a) and (b) according to (11), we now obtain as the approximate solution of the Petrov-Galerkin equations

$$u_n = \frac{1}{2} \sum_{k=1}^{n} c_k \sin \frac{k\pi}{\ell} (x \pm t) , \tag{18}$$

where the c_k are defined above in (9), which tends in the limit to the Fourier series solution for the forward and backward solutions of the plucked string problem given in (8).

If we replace the $\eta(x,t)$ of (17) by

$$\eta(x,t) = \sin \frac{j\pi}{\ell} (x \pm t) \quad , \qquad j = 1, 2, 3, --- , \tag{19}$$

then we obtain on working through the above steps that

$$\int_0^\ell f(x) \cos \frac{k\pi}{\ell} x \, dx = 0 \tag{20}$$

which implies that the coefficients of the Fourier cosine series of $f(z)$ are all zero, and hence, that $f(x \pm t)$ must be treated as an odd periodic function outside the interval $[0,\ell]$. This is consistent with the known structure of the forward and backward components of the solution of the wave equation (see, for example, Lanczos [6], Problem 310).

Finally, we note that, if we replace the $\eta(x,t)$ of (17) and (19) by

$$\eta(x,t) = \cos \frac{j\pi}{\ell} (x \mp t) \quad , \sin \frac{j\pi}{\ell} (x \mp t) \quad , \qquad j = 1, 2, 3, ---$$

we obtain the result that the wave equation (1), (2) and (3) with

$g(x) = 0$ is only solvable, if $f(x) = 0$. This implies that, with respect to the energy norm

$$||u||_*^2 = \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} , u \right) ,$$

the forward and backward components are orthoganal.

References

[1] J. Corones, Bremmer series that correct parabolic approximations, J. Math and Appl. (to appear).

[2] F. W. Sluijter, Generalizations of the Bremmer series based on physical concepts, J. Math and Appl. 27(1969), 282-302.

[3] Jon F. Claerbout, Numerical Holography, chapter 15 in Acoustical Holography, vol. 3, ed. A.F. Metherell, Penum Publ. Corp., New York.

[4] G. N. Yaskova and M. N. Yakovlev, Some conditions for the stability of the method of Petrov-Galerkin (Russia), Trudy Matem. in-ta im. Steklova, 66(1962), 182-189.

[5] M. E. Gurtin, Variational principles for linear initial-value problems, Quart. Appl. Math, 22(1964), 252-256.

[6] C. Lanczos, Linear Differential Operators, van Nostrand, London, 1961.

[7] S. G. Mikhlin, The Numerical Performance of Variational Methods, Walters-Noordhoff, The Netherlands, 1970.